

ENAT-101 - Engineering Calculus

MODULE-1

Real number system :-

Real numbers (R) $\Rightarrow$ Irrational numbers ($Q^c = R - Q$)

$$R = Q \cup Q^c \quad Q^c = \{\sqrt{2}, e, \sqrt{3}, \pi, \dots\}$$

(non-terminating & non-recurring)

$\Rightarrow$ Rational numbers (Q) $\Rightarrow$ Integers (Z)

$$Q = \left\{ \frac{p}{q} = p, q (\neq 0) \in Z \right\} \quad Z = \{-N \cup N \cup 0\}$$

$\Rightarrow$ Whole numbers (W) $\Rightarrow$ Natural numbers (N)

$$W = \{N \cup 0\} \quad N = \{1, 2, 3, 4, \dots\}$$

Archimedean property :-

If n is greater than y ($n, y \in +$ real numbers).

$$n > y$$

then there exist a +ve integer n such that

$$ny > n$$

Intervals :-

• open-intervals $\Rightarrow (a, b) = \{n \in R : a < n < b, a, b \in R\}$

• closed-intervals $\Rightarrow [a, b] = \{n \in R : a \leq n \leq b, a, b \in R\}$

• half-closed-intervals $\Rightarrow [a, b) = \{n \in R : a \leq n < b, a, b \in R\}$
 $(a, b] = \{n \in R : a < n \leq b, a, b \in R\}$

Bounds :-

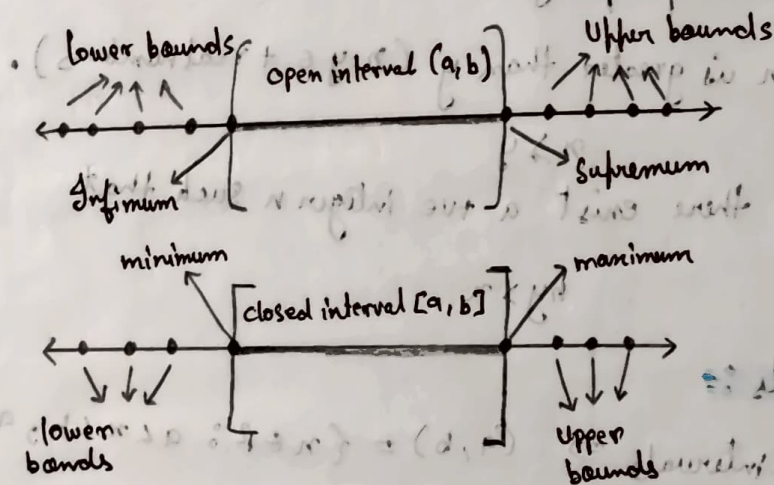
• maximum of set $\Rightarrow$ maximum of a set is largest element of a set (if exists) is called maximum of a set.

• minimum of set $\Rightarrow$ minimum of a set is smallest element of a set (if exists) is called minimum of a set.

- Bounded above $\Rightarrow$ A bounded above set is a set which has a number K which is greater or equal to all numbers in the set. K is called upper bound of the set.
- Bounded below $\Rightarrow$ A bounded below set is a set which has a number k which is smaller than or equal to all numbers in the set. k is called lower bound of the set.
- Bounded $\Rightarrow$ A Bounded set is a set which has both upper and lower bound.

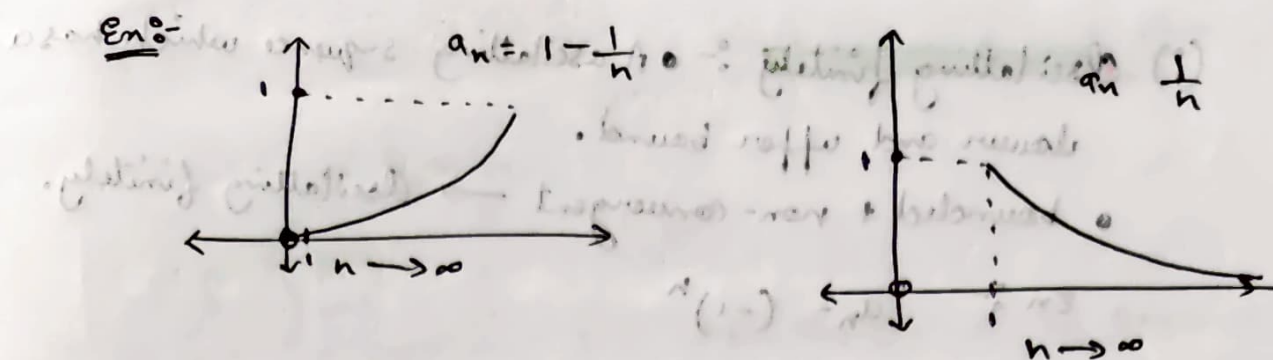
Supremum & Infimum :-

- Supremum is the smallest upper bound of a set.
- Infimum is the largest lower bound of a set.



Convergent sequences :-

- A sequence is said to be convergent when its values approach towards a same number as we approach the n to ∞ .
- * A sequence which is monotonically increasing and bounded above it is convergent and converges at its supremum.
- * A sequence which is monotonically decreasing and is bounded below it is convergent and converges at its infimum.



- Every convergent sequence is bounded.
- Every bounded sequence is not convergent.

Note:- all $\lim$ of n should be unique.

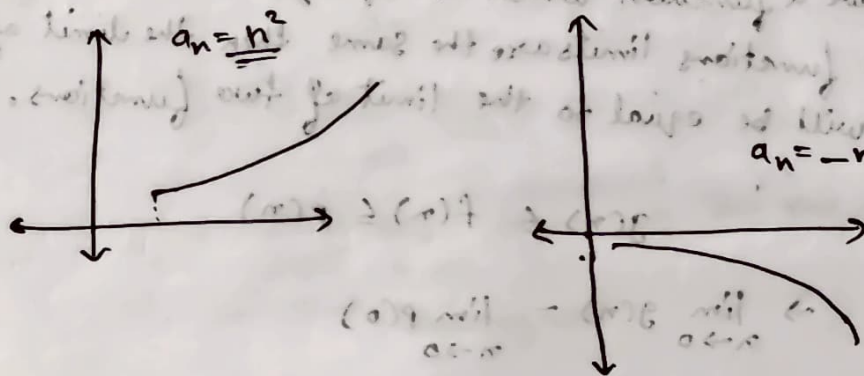
Divergent Sequences:-

- A sequence is said to divergent when its values are approaching either to $+\infty$ and $-\infty$.

* A sequence which is monotonously increasing and unbounded above it diverge towards $+\infty$.

* A sequence which is monotonously decreasing and unbounded below it will diverge towards $-\infty$.

Ex:-



Oscillating Sequences:-

- oscillating sequence is a type of sequence in mathematics where the terms alternate or fluctuates between two distinct values.

* Alternating between positive or negative numbers.

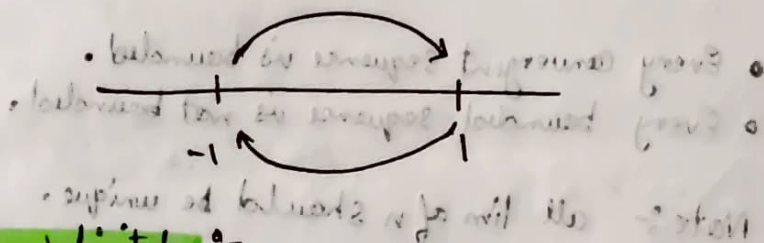
* Displays a periodic or repetitive behaviour.

* They do not have a single limit, as they do not converge to a specific value.

(i) **Oscillating finitely** :- • A oscillating sequence which has a lower and upper bound.

• bounded + non-convergent $\rightarrow$ Oscillating finitely.

Ex :- $a_n = (-1)^n$



(ii) **Oscillating infinitely** :-

• A oscillating sequence which is unbounded and has no upper bound.

• unbounded + non-divergent $\rightarrow$ Oscillating infinitely

Ex :- $a_n = (-1)^n n$

Sandwich theorem :-

(pinching theorem or squeeze theorem)

• If you have a function which is squeezed between two functions, and two functions limits are the same then the limit of squeezed function will be equal to the limit of two functions.

$$g(n) \leq f(n) \leq p(n)$$

$$\Rightarrow \lim_{n \rightarrow \infty} g(n) = \lim_{n \rightarrow \infty} p(n)$$

then

$$\lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} g(n) = \lim_{n \rightarrow \infty} p(n)$$

Ex :- $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = ?$

$\Rightarrow -1 \leq \sin x \leq 1$ [range of sin function]

$\Rightarrow \frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ [divide by n]

Ex: $0 \leq \frac{\sin n}{n} \leq 0$ — Hence proved that $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$.

Ratio test and Root test :-

• Ratio test :-

* The Sequence $\{a_n\}$ where $a_n > 0$ then the sequence will be :

Convergent if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$

Divergent if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1$

Note:- $\lim_{n \rightarrow \infty} a_n = 0$ then it is convergent.

$\lim_{n \rightarrow \infty} a_n > 0$ then it is divergent.

Ex:- $a_n = \frac{2^n}{n!}$, is the sequence convergent or divergent?

$$\Rightarrow * a_n = \frac{2^n}{n!}, * a_{n+1} = \frac{2^{n+1}}{(n+1)!}$$

$$* \frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \times \frac{n!}{2^n} \Rightarrow \frac{2}{n+1}$$

$$* \lim_{n \rightarrow \infty} \frac{2}{n+1} \Rightarrow 0$$

$\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 0$ the Sequence $\{a_n\}$ is Convergent.

• Root test :-

* The Sequence $\{a_n\}$ is a positive term series then it will be:

Convergent if $\lim_{n \rightarrow \infty} (a_n)^{1/n} = d$ and $d < 1$.

Divergent if $\lim_{n \rightarrow \infty} (a_n)^{1/n} = d$ and $d > 1$.

Ex:- $a_n = \left(1 + \frac{1}{n}\right)^{n^2}$, find whether $\{a_n\}$ is convergent or divergent?

$$\Rightarrow a_n = \left(1 + \frac{1}{n}\right)^{n^2}$$

Root test on a_n : $(a_n)^{1/n} = \left[\left(1 + \frac{1}{n}\right)^{n^2}\right]^{1/n}$

$$= \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

$$\Rightarrow \lim_{n \rightarrow \infty} e^{\frac{1}{n} \times n} \left[\left(1 + \frac{1}{n}\right)^n = e^{f(n) \times g(n)} \right]$$

$$\Rightarrow \lim_{n \rightarrow \infty} e = e$$

$\because e$ is greater than 1, $\{a_n\}$ is divergent.

Subsequences :-

- Let $\{a_n\}$ be a sequence and $\{n_1, n_2, \dots\}$ be a sequence of positive integers such that $i > j$ implies $n_i > n_j$. Then the sequence $\{a_{n_i}\}_{i=1}^{\infty}$ is called a subsequence of $\{a_n\}$.

- If sequence is bounded, monotone and convergent then the subsequence will be bounded, monotone and convergent.

Bolzano Weierstrass theorem :-

- Every bounded sequence has a convergent subsequence.

Algebraic properties of convergent sequences :-

- Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences which are convergent.

$$\lim_{n \rightarrow \infty} a_n = L \text{ and } \lim_{n \rightarrow \infty} b_n = M. \text{ Then}$$

$$(i) \lim_{n \rightarrow \infty} (a_n + b_n) = L + M.$$

$$(ii) \lim_{n \rightarrow \infty} (a_n - b_n) = L - M$$

$$(iii) \lim_{n \rightarrow \infty} c a_n = c L.$$

$$(iv) \lim_{n \rightarrow \infty} a_n b_n = L \times M.$$

$$(v) \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{M} \quad (M \neq 0).$$

Cauchy's theorem :-

$$* \text{ If } a_n \rightarrow l$$

$$\Rightarrow \frac{a_1 + a_2 + \dots + a_n}{n} = l \quad [\text{Average of } n \text{ terms}]$$

Series :-

$$\text{let } a_n = n$$

$$\sum_{n=1}^{\infty} a_n = 1 + 2 + \dots + n$$

$\Rightarrow \sum a_n$ is a series

Comparison test for series :-

If $\sum u_n$ & $\sum v_n$ be two series of positive n terms.

$$\text{where } u_n \leq v_n$$

(a) $\sum v_n$ is convergent then $\sum u_n$ is also convergent if

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = K \quad (K \text{ is positive finite number})$$

(b) $\sum v_n$ is divergent then $\sum u_n$ is also divergent if

$$\text{where } u_n \geq v_n \quad \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = K \quad (K \text{ is positive finite number})$$

P-Series test :-

$$\text{The Series } \sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p}$$

is convergent if $p > 1$

is divergent if $p \leq 1$

Conditions for Convergence of Series or Divergence

$$* S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$\{S_n\} \Rightarrow S_1 \leq S_2 \leq S_3 \dots \leq S_n$$

the sequence of sum of terms is monotonically increasing.

if $\{S_n\}$ is bounded above then it is convergent.

if $\{S_n\}$ is unbounded above then it is divergent.

$$* S_n = a_1 + a_2 + a_3 \dots a_n$$

$$\{S_n\} \Rightarrow S_1 \geq S_2 \geq S_3 \dots \geq S_n$$

the sequence of sum of term is monotonically decreasing.

if $\{S_n\}$ is bounded below then it is convergent.

if $\{S_n\}$ is unbounded below then it is divergent.

Lebniz test for alternating Series

* if an alternating series is satisfying (a) & (b) condition

$$\sum_{n=1}^{\infty} (-1)^{n-1} \times u_n \Rightarrow u_1 - u_2 + u_3 - u_4 \dots$$

it is convergent

(a) $u_1 > u_2 > u_3 > u_4 \dots$ The preceding term is numerically smaller than the term.

$$u_{n+1} < u_n$$

(b)

$$\lim_{n \rightarrow \infty} u_n = 0$$

$$\text{Ex :- } \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} \dots - \frac{1}{n!}$$

Prove convergence :-

$\lim_{n \rightarrow \infty} \frac{1}{n!} \Rightarrow 0$ & the numerical value of the term in series is monotonically decreasing.

$\therefore$ By Leibnitz test Series $\frac{1}{1!} + \frac{1}{2!} - \dots - \frac{1}{n!}$ is convergent

Absolute convergence Series

* A series $\sum u_n$ which contains both positive and negative terms is said to be absolutely convergent if both $\sum u_n$ & $\sum |u_n|$ is convergent

Ex :- $\sum u_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \dots - (-1)^{n-1} \times \frac{1}{n}$

$\sum u_n$ is convergent by Leibnitz test

$\Rightarrow \sum |u_n| = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \dots - \frac{1}{n}$

$\sum |u_n|$ is monotonically decreasing — (1)

$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ — (2)

by (1) & (2)

$\sum |u_n|$ is convergent

Limit

existence of a limit at a point is when LHL and RHL both exist and are equal and common value is called limit of the function at a given point.

$$\begin{array}{c} \text{LHL} = \text{RHL} = \lim_{n \rightarrow a} f(n) \\ \swarrow \quad \searrow \\ \lim_{n \rightarrow a-h} f(n) \quad \lim_{n \rightarrow a+h} f(n) \\ * [h \rightarrow 0] \end{array}$$

* Algebra of limits

$\lim_{n \rightarrow a} [f(n) \pm g(n)] = l \pm m$

$$(2) \lim_{x \rightarrow a} f(x)g(x) = lm$$

$$(3) \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{l}{m}, \text{ [provided } m \neq 0]$$

Ex:-

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x}$$

$$\text{LHL} \Rightarrow \lim_{x \rightarrow 0-h} x \sin \frac{1}{x}$$

$$= \lim_{h \rightarrow 0} (0-h) \times \sin \left(\frac{1}{0-h} \right)$$

$$= \lim_{h \rightarrow 0} -h \times \sin \left(\frac{1}{h} \right)$$

$$= \lim_{h \rightarrow 0} h \times \sin \left(\frac{1}{h} \right)$$

$$= \underline{0} \quad [\sin a \in [-1, 1]]$$

$$\text{RHL} \Rightarrow \lim_{x \rightarrow 0+h} x \sin \frac{1}{x}$$

$$= \lim_{h \rightarrow 0} (0+h) \times \sin \left(\frac{1}{0+h} \right)$$

$$= \lim_{h \rightarrow 0} h \times \sin \frac{1}{h}$$

$$= \underline{0}$$

$$\therefore \text{LHL} = \text{RHL} = 0 \Rightarrow \lim_{x \rightarrow 0} \sin \frac{1}{x} \times x = 0$$

* Some Important property

$$\text{If } |f(x)| \leq k|x| \quad \forall x \in \mathbb{R}$$

$$\text{then } \lim_{x \rightarrow 0} f(x) = 0$$

Ex:-

$$|f(x)| = \left| \sin x \cos \frac{1}{x} \right|$$

$$\Rightarrow \left| \sin x \cos \frac{1}{x} \right| \leq |\sin x| \quad \left[\cos \frac{1}{x} \in [-1, 1] \right]$$

$$\Rightarrow \left| \sin x \cos \frac{1}{x} \right| \leq |x| \quad [x \geq \sin x]$$

$$\therefore \left| \sin x \cos \frac{1}{x} \right| \leq k|x| \text{ where } k < 1$$

$$\lim_{x \rightarrow 0} \cos \frac{1}{x} \times \sin x = 0$$

$$\lim_{x \rightarrow 0} x^\alpha \times \sin \frac{1}{x^\beta} = 0, \alpha > 0$$

$$\lim_{x \rightarrow 0} x^\alpha \times \cos \frac{1}{x^\beta} = 0, \alpha > 0$$

Continuity of a function

* Cauchy's definition on continuity

$$\lim_{x \rightarrow a} f(x) \text{ where } x \in (a-\delta, a+\delta)$$

$$\Downarrow$$

$$h \text{ where } h \in (f(a)-\epsilon, f(a)+\epsilon)$$

limits of $f(x)$ where $x \rightarrow$ values in range of $(a-\delta, a+\delta)$ should belong to $(f(a)-\epsilon, f(a)+\epsilon)$.

* Arithmetical definition of continuity

$$x=a, \lim_{x \rightarrow a} f(x) = f(a)$$

$$\text{if } \lim_{x \rightarrow a-h} f(x) = \lim_{x \rightarrow a+h} f(x) = \lim_{x \rightarrow a} f(x)$$

$$\text{or}$$

$$\text{LHL} = \text{RHL} = f(a)$$

A function is said to be continuous if LHL and RHL are equal and both of them are equal to $f(a)$.

* Continuity in Interval :- A function is said to be continuous in an interval if it is continuous in whole range or all values of interval.

Ex :- $f(x)$ is continuous in an interval $[a, b]$ if it is continuous in all values where $x \in [a, b]$.

* Sequential definition of Interval :- A function f defined on the interval I is said to be continuous at $x=a$, where $a \in I$. If for every sequence $\langle a_n \rangle$ converging to $(a_n \rightarrow a)$, every sequence $\langle f(a_n) \rangle$ converges $f(a)$.

i.e. for every $a_n \rightarrow a$, then $f(a_n) \rightarrow f(a)$

Discontinuity of a function

→ A function f which is not continuous at a point ' a ' is said to be discontinuous at the point ' a '.

' a ' is called the point of discontinuity of function f or f is said to have a discontinuity at a .

→ A function which is discontinuous even at a single point of an interval is said to be discontinuous in the interval.

Types of discontinuity :-

(i) Removable Discontinuity.

(ii) Discontinuity of first kind (OR Jump Discontinuity).

(iii) Discontinuity of second kind.

(iv) Mixed Discontinuity.

(i) Removable discontinuity $\Rightarrow$

$$\nexists \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) \neq f(a)$$

if LHL is equal to RHL but is not equal to $f(a)$, this kind of discontinuity at ' a '.

$$\text{Ex :- } f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(ii) Discontinuity of first kind $\Rightarrow$

$\nexists \lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ both exist but are not equal ($LHL \neq RHL$).

Then $f(x)$ is said to have discontinuity of first kind at ' a '.

$$\text{Ex :- } f(x) = [x]$$

$$f(x) = \{x\}$$

(iii) Discontinuity of Second kind $\Rightarrow$

If Neither $\lim_{n \rightarrow a^-} f(n)$ nor $\lim_{n \rightarrow a^+} f(n)$ exist, then $f(n)$ is said to have a discontinuity of second kind at 'a'.

Ex:-

$$f(n) = \sin \frac{1}{n}, n \neq 0$$

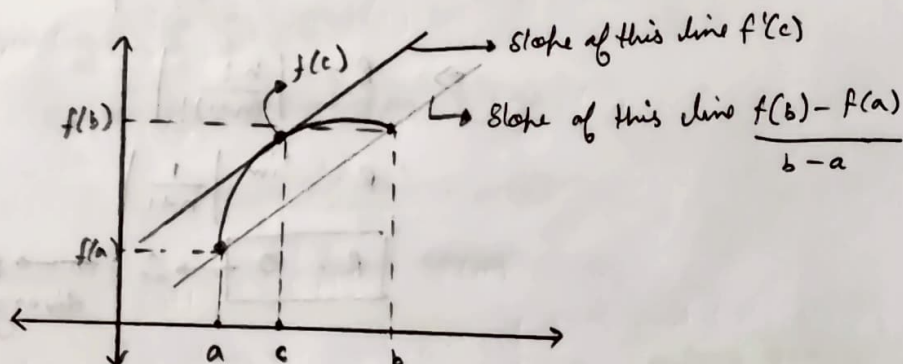
(iv) Mixed Discontinuity $\Rightarrow$

If in a function $f(n)$ LHL or RHL exist but another doesn't it is termed as mixed discontinuity.

$$\text{Ex:- } f(n) = \begin{cases} \sin \frac{1}{n}, & n < 0 \\ n, & n \geq 0 \end{cases}$$

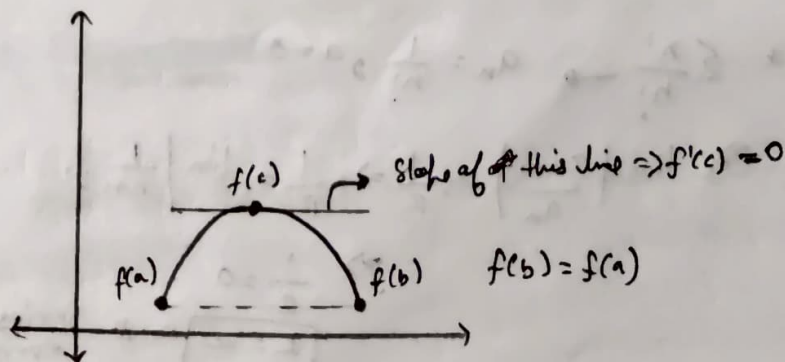
Mean Value Theorem

According to MVT if a function $f(n)$ is continuous on $[a, b]$ and differentiable on (a, b) , then there exist a number or point C ($C \in (a, b)$) where $f'(C) = \frac{f(b) - f(a)}{b - a}$.



Rolle's Theorem

According to Rolle's Theorem if a function $f(n)$ is continuous and on $[a, b]$ and differentiable on (a, b) and if $f(b) = f(a)$ then a exist at least one point in (a, b) where $f'(C) = 0$.



Power Series

$$\sum_{n=0}^{\infty} C_n (x-a)^n \rightarrow \text{A power series centered at } a$$

$$\Rightarrow \sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \dots$$

Power Series will be convergent for a radius of convergence

$$\Rightarrow |x-a| < R \rightarrow R \text{ is radius of convergence [convergent]}$$

$$|x-a| > R \rightarrow \text{Divergent}$$

To find radius of convergence :-

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}$$

Ex:-

* $\sum n! x^n \rightarrow$ power series where $a=0$, $a_n = n!$

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \Rightarrow \frac{1}{R} = \left| \frac{(n+1)!}{n!} \right|$$

$$\Rightarrow R = \left| \frac{n!}{(n+1)!} \right|$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right|$$

$$\boxed{R=0} \rightarrow \therefore R=0 \text{ series divergent everywhere}$$

$$* \sum \frac{x^n}{n!} \rightarrow a_n = \frac{1}{n!}, a=0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R} \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = \frac{1}{R}$$

$$\Rightarrow \frac{1}{R} = 1 \Rightarrow \boxed{R=1} \rightarrow |x-0| < 1 \text{ for convergence}$$

$$* \sum \frac{x^n}{n!} \rightarrow a_n = \frac{1}{n!}, a=0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R} \Rightarrow \frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right|$$

$$\Rightarrow \frac{1}{R} = 0$$

Taylor's Series

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$

$$f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \dots$$

$$* f(a) = C_0 \text{ --- (1)}$$

$$f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \dots$$

$$* f'(a) = C_1 \text{ --- (2)}$$

$$f''(x) = 2C_2 + 6C_3(x-a) + \dots$$

$$\Rightarrow f''(a) = 2C_2$$

$$* \frac{f''(a)}{2} = C_2 \text{ --- (3)}$$

$$f'''(x) = 6C_3(x-a) + \dots$$

$$f'''(a) = 6C_3$$

$$* \frac{f'''(a)}{6} = C_3 \text{ --- (4)}$$

By observing (1), (2), (3) & (4)

$\Rightarrow a_n$ can be represented as $\frac{f^n(a)}{n!}$

$$C_n = \frac{f^n(a)}{n!} \text{ --- (5)}$$

Replacing (5) in C_n of power series

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x-a)^n \longrightarrow \text{Taylor's Series}$$

Maclaurin Series

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \longrightarrow \text{Taylor series where } a=0$$

Application of Taylor Series

Taylor series can be used to convert some functions in form of series.

$f(x) = e^x$, To convert e^x in series [Taylor series $a=0$]

$$\begin{aligned} \Rightarrow f(0) &= 1 \\ f'(0) &= 1 \\ f''(0) &= 1 \\ &\vdots \\ f^n(0) &= 1 \end{aligned} \Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\Rightarrow e^x = \frac{1}{1} + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$f(x) = \sin(x)$

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= \cos(0) = 1 \\ f''(0) &= -\sin(0) = 0 \\ f'''(0) &= -\cos(0) = -1 \\ f^{(4)}(0) &= \sin(0) = 0 \\ f^{(5)}(0) &= \cos(0) = 1 \end{aligned}$$

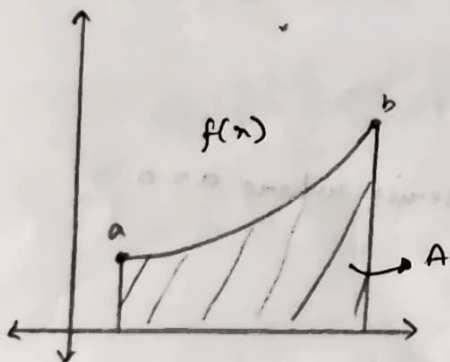
$\sin(x) =$

$$0 + 1x + 0x^2 + \frac{-1}{3!}x^3 + 0x^4 + \dots$$

$$\therefore \sin x = \frac{x}{1!} - \frac{1}{3!}x^3 + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$$

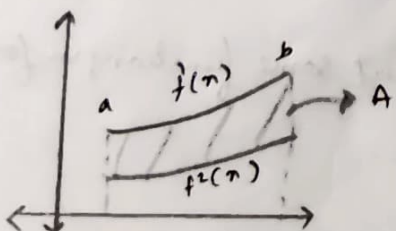
$$\sin(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \times (-1)^n$$

Integration



$$\int_a^b f(x) dx = A$$

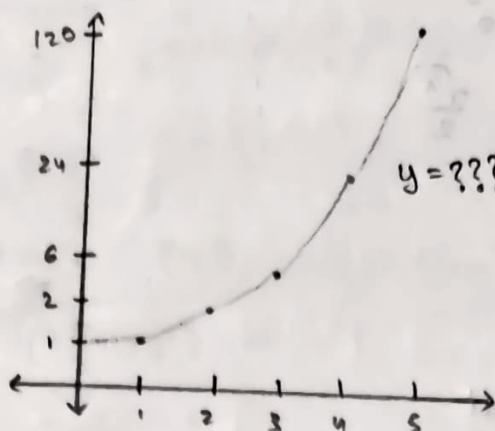
Area the area enclosed between x -axis and $f(x)$ from a to b .



$$A \Rightarrow \int_a^b [f_1(x) - f_2(x)] dx$$

The Gamma Function

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120



Important

$$(n+1)! = (n+1)n!$$

or

$$n! = n(n-1)!$$

Gamma function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

$$\textcircled{1} - \Gamma(1) = \int_0^{\infty} t^0 e^{-t} dt = \int_0^{\infty} e^{-t} dt = 1 \Rightarrow 0!$$

$$\textcircled{2} - \Gamma(2) = \int_0^{\infty} t^1 e^{-t} dt = \int_0^{\infty} t e^{-t} dt = 1 \Rightarrow 1!$$

$$\textcircled{3} - \Gamma(3) = \int_0^{\infty} t^2 e^{-t} dt = \left[-(t^2 + 2t + 2) e^{-t} \right]_0^{\infty} = 2 \Rightarrow 2!$$

$$\textcircled{4} - \Gamma(4) = \int_0^{\infty} t^3 e^{-t} dt = \left[-(t^3 + 3t^2 + 6t + 6) e^{-t} \right]_0^{\infty} = 6 \Rightarrow 3!$$

from $\textcircled{1}, \textcircled{2}, \textcircled{3}$ & $\textcircled{4}$

$$\Gamma(z+1) = (z)! \quad \text{or} \quad \Gamma(z) = (z-1)!$$

$$\therefore n! = n(n-1)!$$

$$\Gamma(z+1) = z \Gamma(z)$$

Application of Gamma function

$$\left(\frac{1}{2}\right)! = ??$$

$$\text{let } \frac{1}{2} = z, (z)! = \Gamma(z+1), \Gamma\left(\frac{3}{2}\right) = \left(\frac{1}{2}\right)!$$

$$\Gamma\left(\frac{3}{2}\right) \Rightarrow \int_0^{\infty} t^{\frac{1}{2}} e^{-t} dt$$

$$\Rightarrow \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \times \int_0^{\infty} t^{-1/2} e^{-t} dt$$

$$= \frac{1}{2} \times \int_0^{\infty} \frac{1}{\sqrt{t}} e^{-t} dt$$

$$\Rightarrow \int_0^{\infty} \frac{1}{\sqrt{t}} e^{-t} dt$$

$$\Rightarrow \int_0^{\infty} \frac{1}{\sqrt{t}} e^{-t} dt$$

$$\text{let } u = \sqrt{t} \Rightarrow du = \frac{1}{2\sqrt{t}} dt$$

$$u^2 = t$$

$$\Rightarrow \int_0^{\infty} 2 e^{-u^2} du \Rightarrow 2 \int_0^{\infty} e^{-u^2} du$$

↳ Gaussian Integral

$$\Rightarrow \sqrt{\pi}$$

$$\therefore \Gamma\left(\frac{1}{2}+1\right) = \frac{\sqrt{\pi}}{2} \quad \underline{\underline{\text{Ans}}}$$

Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx \rightarrow \text{Gaussian Integral}$$

$$\Rightarrow G = \int_{-\infty}^{\infty} e^{-x^2} dx, \quad \int_{-\infty}^{\infty} e^{-y^2} dy = G$$

$$\Rightarrow G^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \wedge \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$\Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy$$

$$\Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy$$

$$[x^2 + y^2 = r^2]$$

$$2\pi \int_0^{\infty} r e^{-r^2} dr \Rightarrow r dr d\theta = dx dy$$

$$\int_0^{\infty} \int_0^{2\pi} r e^{-r^2} dr d\theta \Rightarrow \pi \Rightarrow \boxed{G = \sqrt{\pi}}$$

Beta function

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

$$x, y \in \mathbb{C}$$

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} = \int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta \quad \rightarrow \text{Walli's formula}$$

$$= \frac{\Gamma\left(\frac{m'+1}{2}\right) \Gamma\left(\frac{n'+1}{2}\right)}{2 \Gamma\left(\frac{m'+n'+2}{2}\right)} = \frac{1}{2} B\left(\frac{m'+1}{2}, \frac{n'+1}{2}\right)$$

$\Rightarrow$ Different forms of Beta function

(i) $\int_0^1 t^{x-1} (1-t)^{y-1} dt$ $[B(x, y)]$

(ii) $\frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$

(iii) $\int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta$ $[P, Q] \Rightarrow$ Walli's formula

$$= \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2 \times \Gamma\left(\frac{p+q+2}{2}\right)} = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Improper Integrals

* Proper Integrals are the integrals with bounded limits & bounded value of Integrand in whole range of given limits.

$$\int_a^b x^2 dx \rightarrow \text{bounded} = (a, b), \quad x^2 \text{ bounded in } (a, b).$$

* Improper Integrals are integrals where either the limits are unbounded or value of Integrand is unbounded in range of limits.

$$\int_0^{\infty} x^2 dx, \int_0^1 \frac{1}{x} dx, \int_0^{\infty} \frac{1}{x^{2+1}} dx, \int_1^{\infty} \frac{1}{x^p} dx \dots \text{etc}$$

Improper Integral of First Kind

* In this kind of improper integral the limits are unbounded.

$$\Rightarrow \int_0^{\infty} \frac{1}{x^2} dx, \int_0^{\infty} \frac{1}{x^{2+1}} dx, \int_1^{\infty} \frac{1}{x^p} dx \dots \text{etc}$$

* Ways to check converges of Improper Integral First Kind

$\Rightarrow$ P-Test :-

$$\int_1^{\infty} \frac{1}{x^p} dx \rightarrow \begin{cases} \text{converges for } p > 1 \\ \text{diverges for } p \leq 1 \end{cases}$$

Comparison Test :-

Suppose $0 \leq f(x) \leq g(x)$ for $x \geq a$, then

$\int_a^{\infty} f(x) dx$ converges if $\int_a^{\infty} g(x) dx$ converges

$\int_a^{\infty} g(x) dx$ diverges if $\int_a^{\infty} f(x) dx$ diverges

Improper Integral of Second Kind

* In this kind of improper Integral the limits are bounded but value of Integrand is not bounded in range of the limits.

$$\int_a^b \frac{1}{(x-a)^p} dx, \int_0^1 \frac{1}{x} dx \dots \text{etc}$$

P-Test :-

$$\int_a^b \frac{1}{(x-a)^p} dx \rightarrow \begin{cases} \text{converges for } p < 1 \\ \text{diverges for } p \geq 1 \end{cases}$$