

Methods of Getting Soln

1b) (a) $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \quad \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$$\begin{aligned} x &= X+h \\ y &= Y+k \end{aligned} \Rightarrow \frac{dy}{dx} = \frac{a_1 X + b_1 Y + (a_1 h + b_1 k + c_1)}{a_2 X + b_2 Y + (a_2 h + b_2 k + c_2)} \left. \begin{array}{l} \text{Equate} \\ \text{them} \\ \text{to} \\ \text{zero} \end{array} \right\}$$

then solve for the eqn.

(b) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

$$\Rightarrow \frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{d_1 (a_1 x + b_1 y) + c_2} \quad \text{let } t = a_1 x + b_1 y$$

$$\frac{dt}{dx} = a_1 + b_1 \frac{dy}{dx} \quad \text{--- (1)}$$

② - $\frac{dy}{dx} = \frac{t + c_1}{d_1 t + c_2}$ use eqn ① & ② solve for solution.

(c) $a_1 = b_2$ & $b_1 = a_2$

$$\Rightarrow \frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{b_1x + a_1y + c_2} \quad \text{--- (1)}$$

$$\Rightarrow \frac{dy + dn}{dy - dn} = \frac{\text{Eqn (1)} + \text{Eqn (2)}}{\text{Eqn (1)} - \text{Eqn (2)}}$$

$$\Rightarrow \frac{d(y+n)}{d(y-n)} = \frac{q_3(n+y) + c_3}{a_4(y-n) + c_4} \quad \begin{matrix} y+n=t \\ y-n=w \end{matrix}$$

Solve $\frac{dt}{dw} = \frac{a_3 t + c_3}{a_4 w + c_3}$ — Solve

$$2b(a) \quad Mdx + Ndy = 0$$

$$M \& N = \text{function of } (x, y)$$

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then Soln $\Rightarrow \int_{y-\text{const}}^{} M dx + \int_{x-\text{const}}^{} N dy = C$

(b) $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ for $Mdx + Ndy = 0$ — (1)

to get integrating factor

→ (i) $\frac{M_y - N_x}{N} = \phi(x)$ — only in x
then I.F. = $e^{\int \phi(x) dx}$

or

(ii) $\frac{N_x - M_y}{M} = \phi(y)$ — only in y
then I.F. = $e^{\int \phi(y) dy}$

Multiply I.F. to (1) then find General solution.

3/b Linear D.E :-

(a) $\frac{dy}{dx} + P(x)y = Q(x)$ | $\frac{dx}{dy} + P(y)x = Q(y)$

→ I.F. = $e^{\int P(x) dx}$

→ I.F. = $e^{\int P(y) dy}$

Soln → $y \cdot \text{I.F.} = \int Q(x) \cdot \text{I.F.} dx + C$ | Soln → $x \cdot \text{I.F.} = \int Q(y) \cdot \text{I.F.} dy + C$

4/b Solving Higher order Linear Differential Eqn

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + a_3 \frac{d^{n-3} y}{dx^{n-3}} + \dots = V(x)$$

Higher
Order
LDE

≠ $\int V(x) = 0$

→ $y = y_c \rightarrow$ Solution

Method to find y_c

⇒ (i) Find Auxiliary Eqn

⇒ Substitute $\frac{d^n y}{dx^n} \Rightarrow D^n$

then

$D \Rightarrow m$

then find solution for the Eqn (polynomial you got)

$h_1, h_2, h_3 \Rightarrow$ roots of Auxiliary Eqn.

* m values are distinct $m \in (h_1, h_2, \dots)$

$$\Rightarrow y_c = c_1 e^{h_1 n} + c_2 e^{h_2 n} + \dots$$

* m values are repetitive $m \Rightarrow h_1 = h_2 = h_3$

$$\Rightarrow y_c = (c_1 + c_2 n) e^{h_1 n} + c_3 e^{h_2 n}$$

* m values are complex $\Rightarrow \alpha \pm i\beta$ $\rightarrow$ m values are $\alpha \pm i\beta$ & $\alpha \pm i\gamma$

$$\Rightarrow y_c = e^{\alpha n} (c_1 \cos \beta n + c_2 \sin \beta n) \Rightarrow y_c = e^{\alpha n} [(c_1 + c_2 n) \cos \beta n + (c_3 + c_4 n) \sin \beta n]$$

$\neq$ if $v(n) \neq 0$

$$y = y_c + y_p \rightarrow \text{solution}$$

* methods to find y_p

(i) if $v(n)$ is of type e^{an}

$$\Rightarrow y_p = \frac{1}{f(D)} e^{an} \Rightarrow \frac{1}{f(a)} e^{an} \quad (f(a) \neq 0)$$

$$\text{else } \frac{n e^{an}}{f'(D)} \Rightarrow \frac{n e^{an}}{f''(D)} \quad (\text{if } f'(a) = 0 \text{ and so on.})$$

same goes for $c e^{an}$, $c_1 e^{a_1 n} + c_2 e^{a_2 n} \dots$

(ii) if $v(n)$ is of type $\sin(an)$ / $\cos(an)$

$$\Rightarrow y_p = \frac{1}{f(D^2)} \sin an$$

$$\Rightarrow \frac{1}{f(-a^2)} \sin an \quad [\text{substitute } D^2 = -a^2]$$

if there are terms of D in denominator do the following

$$(i) \frac{u \sin n}{D^4 + 2D^3 - 3D^2} \Rightarrow \frac{u \sin n}{(-1)^2 + 2 \times (-1)^3 - 3 \times (-1)^2}$$

$$\Rightarrow \frac{2}{5} (\sin n (2+D)) \quad | \Rightarrow \frac{u \sin n}{1 - 2D + 3}$$

$$\Rightarrow \frac{2}{5} \left[2 \sin n + \frac{d(\sin n)}{dn} \right] \quad | \Rightarrow \frac{u \sin n}{4 - 2D} \Rightarrow \frac{2 \sin n}{2 - D}$$

$$\Rightarrow \frac{2}{5} [2 \sin n + \cos n] \quad | \Rightarrow \frac{2 \sin n (2+D)}{(4 - D^2)} \Rightarrow \frac{2 \sin n (2+D)}{4 - (-1)} \Rightarrow \frac{2}{5} \sin n (2+D)$$

* if P is in Denominator then integrate only $\sin x$ or $\cos x$

Picard's Uniqueness & Existence Theorem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

$$R \Rightarrow |x - x_0| \leq a, \quad |y - y_0| \leq b$$

Conditions $\Rightarrow$

(1) On R $f(x, y)$ is continuous.

(2) $f(x, y)$ is bounded on R

$$|f(x, y)| \leq M$$

(3) f satisfies:-

$$|f(x, y_1) - f(x, y_2)| \leq K |y_1 - y_2|$$

$$\text{or} \quad \left| \frac{\partial f(x, y)}{\partial y} \right| \leq K$$

$\Rightarrow \exists$ a unique soln in $|x - x_0| \leq h$

$$h = \min(a, b/M)$$

Reduction of Order

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0 \quad y_1 \text{ is given}$$

$$y_2 = y_1 \int \frac{e^{-\int P(x) dx}}{(y_1(x))^2} dx$$

Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Method of Variation of Parameters

$$\# a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x) \text{ --- Given}$$

$$\# \text{ let } y_c = C_1 y_1(x) + C_2 y_2(x)$$

$$y_p \Rightarrow v_1(x)y_1(x) + v_2(x)y_2(x)$$

$$\Rightarrow v_1(x) = - \int \frac{f(x)y_2(x) dx}{a_0(x)w(y_1, y_2)} \quad v_2(x) = \int \frac{f(x)y_1(x) dx}{a_0(x)w(y_1, y_2)}$$

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$