

Vector algebraDot product ($\vec{A} \cdot \vec{B}$)

$$\Rightarrow |\vec{A} \cdot \vec{B}| = |\vec{A}| |\vec{B}| \cos \theta \quad \star$$

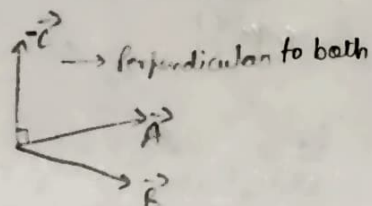
$$|\vec{A} \cdot \vec{B}| = (a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}) \cdot (a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}) \quad \star$$

$$|\vec{A} \cdot \vec{B}| = a_1 a_2 + b_1 b_2 + c_1 c_2$$

Cross product ($\vec{A} \times \vec{B}$)

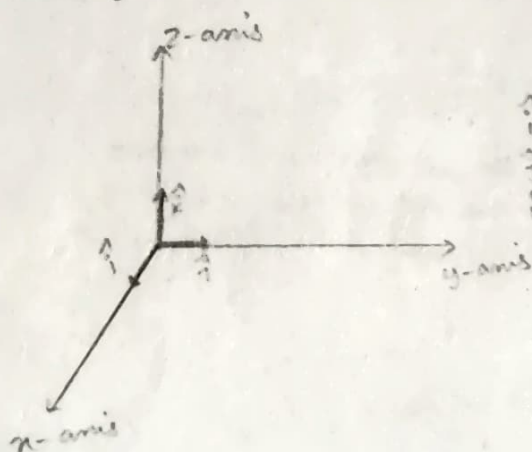
$$\Rightarrow |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta \quad \star$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \vec{C} \quad \star$$

# Coordinate systemCartesian coordinate system

In every coordinate system there are unit vectors, by which we can create every vector in the space. For every different coordinate system there are different unit vectors.

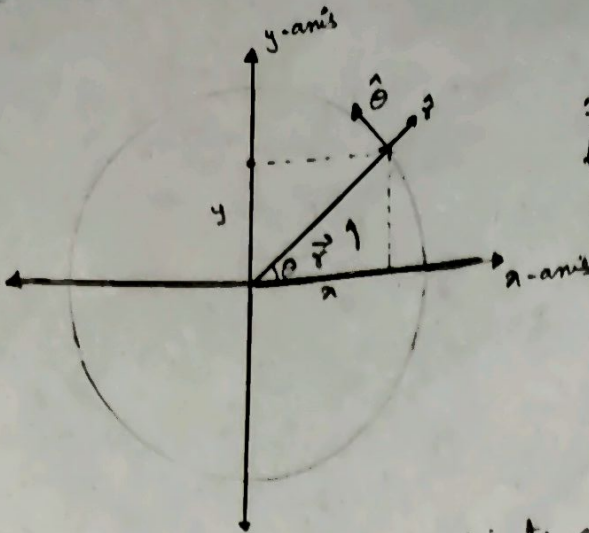
$\hat{i} + \hat{j} + \hat{k} \rightarrow$ unit vectors of Cartesian coordinate system.



$\hat{i} \rightarrow$ unit vector in direction x-axis.
 $\hat{j} \rightarrow$ unit vector in direction y-axis.
 $\hat{k} \rightarrow$ unit vector in direction z-axis.

Rahul

Polar Coordinate System (2D)



$\hat{r}$ is the unit vector for $\vec{r}$.
 $\hat{\theta}$ is the unit vector for θ .

Unit Vectors in 2D polar coordinate system

$$\vec{r} = \frac{\partial \vec{r}}{\partial r} \Rightarrow \vec{r} = x\hat{i} + y\hat{j} = r\cos\theta\hat{i} + r\sin\theta\hat{j} \quad \text{--- (1)}$$

$$\frac{\partial \vec{r}}{\partial r} = \cos\theta\hat{i} + \sin\theta\hat{j}, \quad \left| \frac{\partial \vec{r}}{\partial r} \right| = 1$$

$$\therefore \hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j} \quad \text{--- *}$$

$$\hat{\theta} = \frac{\partial \vec{r}}{\partial \theta} \Rightarrow \vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$$

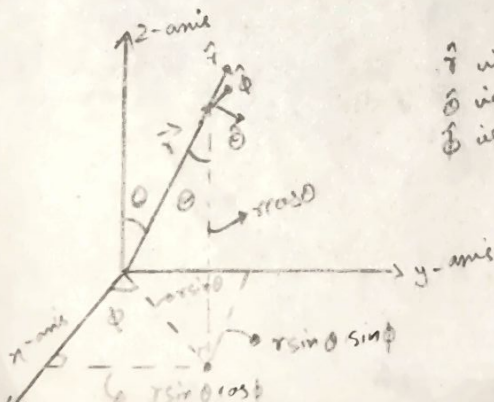
$$\frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\hat{i} + r\cos\theta\hat{j}$$

$$\left| \frac{\partial \vec{r}}{\partial \theta} \right| = r$$

$$\therefore \hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j} \quad \text{--- *}$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x} \quad \text{--- *}$$

Polar Coordinate System (3D)



$\hat{r}$ is the unit vector in $\vec{r}$ direction
 $\hat{\theta}$ is the unit vector in θ direction
 $\hat{\phi}$ is the unit vector in ϕ direction

Amal

$$\vec{r} = r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} + r \cos \theta \hat{k}$$

Unit vectors in 3D polar coordinate system

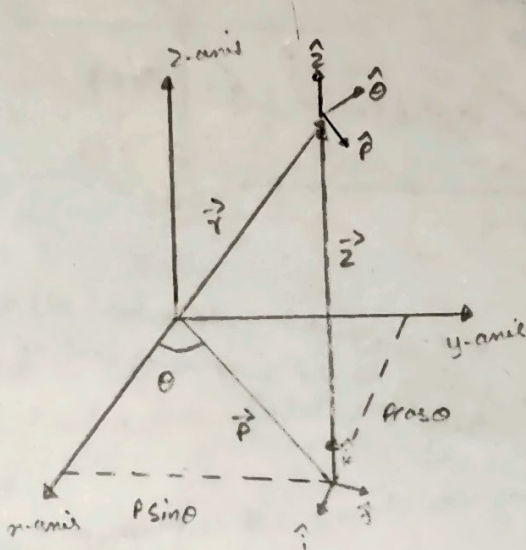
$$\Rightarrow \hat{r} = \frac{\frac{\partial \vec{r}}{\partial r}}{\left| \frac{\partial \vec{r}}{\partial r} \right|} = \frac{\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}}{1}$$

$$\hat{\theta} = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} \rightarrow *$$

$$\hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left| \frac{\partial \vec{r}}{\partial \phi} \right|} = -\sin \phi \hat{i} + \cos \phi \hat{j} \rightarrow *$$

$$\hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left| \frac{\partial \vec{r}}{\partial \phi} \right|} = -\sin \phi \hat{i} + \cos \phi \hat{j} \rightarrow *$$

Cylindrical Coordinate System



$\hat{r}$ is unit vector in $\vec{r}$ direction
 $\hat{z}$ is unit vector in $\vec{z}$ direction
 $\hat{\phi}$ is unit vector in ϕ direction

$$\vec{r} = \rho \cos \phi \hat{i} + \rho \sin \phi \hat{j} + z \hat{k}$$

$$\hat{r} = \frac{\frac{\partial \vec{r}}{\partial \rho}}{\left| \frac{\partial \vec{r}}{\partial \rho} \right|} = \cos \phi \hat{i} + \sin \phi \hat{j} \rightarrow *$$

$$\hat{z} = \frac{\frac{\partial \vec{r}}{\partial z}}{\left| \frac{\partial \vec{r}}{\partial z} \right|} = \hat{k} \rightarrow *$$

$$\hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left| \frac{\partial \vec{r}}{\partial \phi} \right|} = -\sin \phi \hat{i} + \cos \phi \hat{j} \rightarrow *$$

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2}, & x &= \rho \cos \phi \rightarrow * \\ \phi &= \tan^{-1} \frac{y}{x}, & y &= \rho \sin \phi \rightarrow * \\ z &= z, & z &= z \rightarrow * \end{aligned}$$

Gradient, Divergence and Curl

$\vec{\nabla}$ del operator is a vector differential equation that operates on scalar and vector.

In different coordinate system it has different equation

(Cartesian coordinate system)

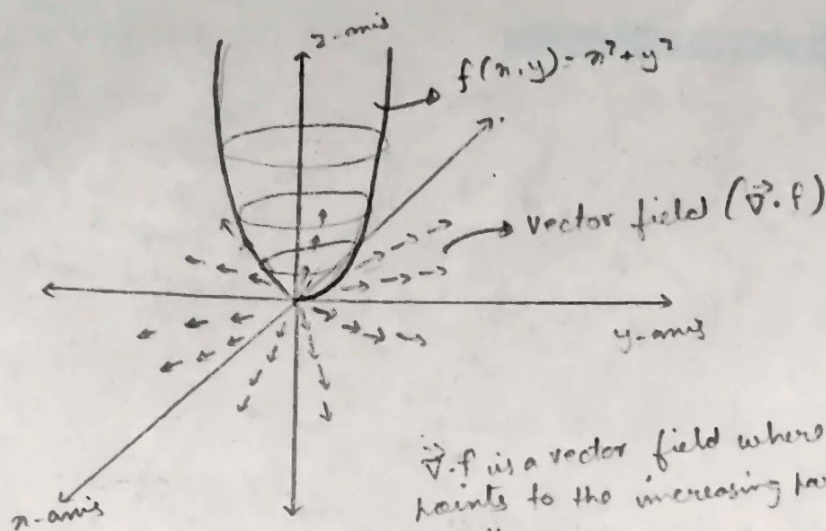
$$\Rightarrow \vec{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \quad \text{---} \star$$

Gradient

Gradient is the product of a scalar quantity and $\vec{\nabla}$ operator. This gives a vector field directing to the steepest increase.

$$\vec{\nabla} f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \quad \text{---} \star \quad \left[f = x+y+z \text{ or } f = x^2+y^2 \right]$$

Example



$\vec{\nabla} \cdot f$ is a vector field where all vectors point to the increasing part of f , down as well.

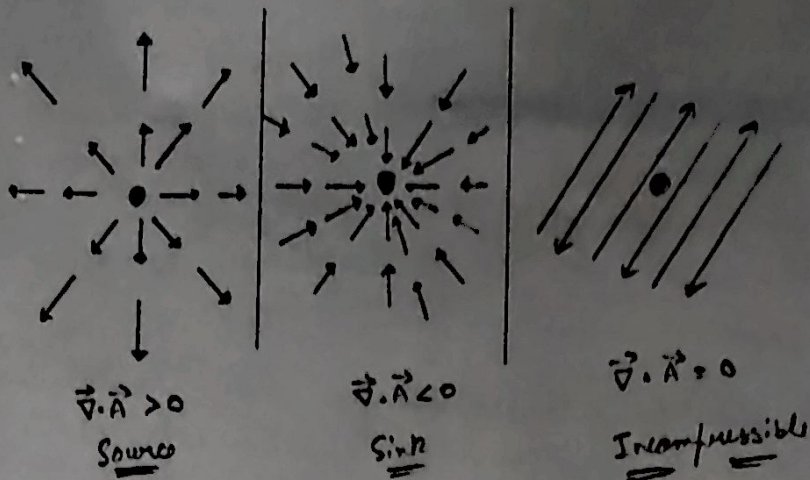
Divergence

Divergence of a vector is dot product of del operator and the vector. This gives the tendency of the vector field to disperse or collect at a given point.

$$\nabla \cdot \vec{A} = \frac{\partial \vec{A}_x}{\partial x} + \frac{\partial \vec{A}_y}{\partial y} + \frac{\partial \vec{A}_z}{\partial z} \quad \left[\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z \right]$$

[Only for Cartesian plane]

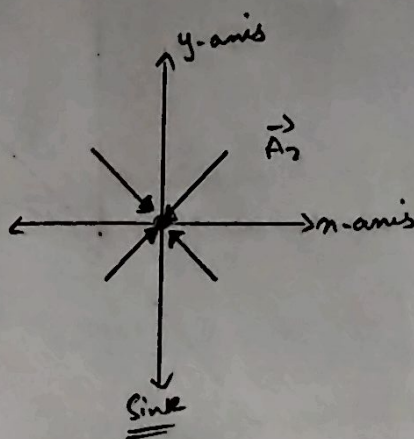
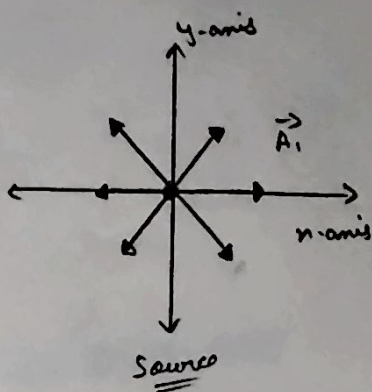
Prof



Example :-

Source $\Rightarrow \vec{A}_1(x, y) = x\hat{i} + y\hat{j}$

Sink $\Rightarrow \vec{A}_2(x, y) = -x\hat{i} - y\hat{j}$



Curl

Curl of a vector field is cross product of del operator and the vector field.

Curl quantifies the circulation of a vector field.

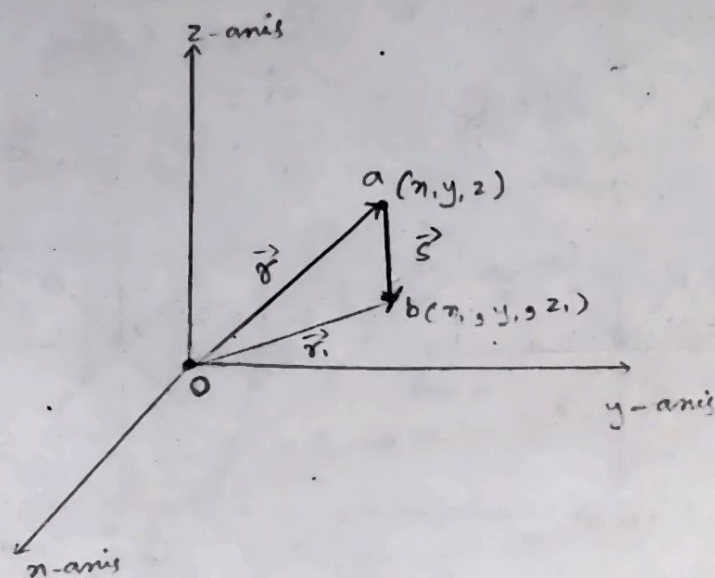
$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{i} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{k}$$

[Only for Cartesian plane]

Position vectors and Displacement

Every thing in kinematics is about the change of position, change of velocity etc. These all quantities are quantified in vector forms.

Vectors quantifying the position of an object in accordance to a reference frame is known as position vector ($\vec{r}$). We can imagine it in a 3D space using Cartesian coordinate system.



In this coordinate system O is frame of reference, we can set our observer sitting at origin. There is a point 'a' with coordinates (x, y, z) . We can say that there is an object at that point 'a' and vector which quantifies the object's position in position vector ' $\vec{r}$ '.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

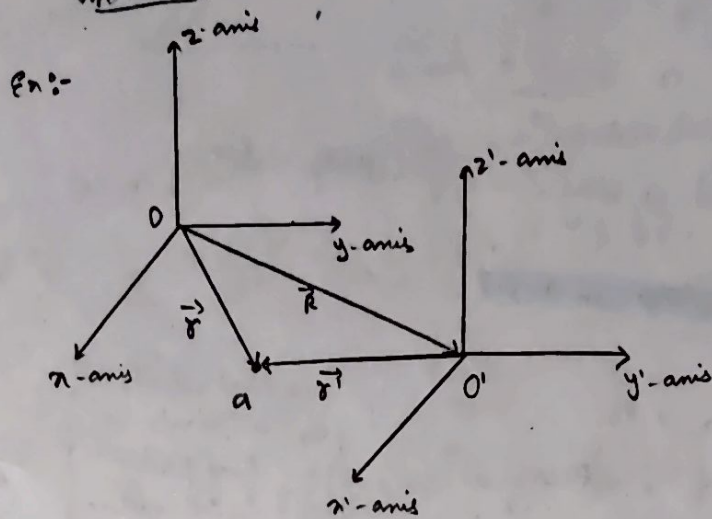
Displacement is the change in position vector. We can say that our object has moved from a to b -position then displacement will be

$$\vec{s} = \vec{r}_2 - \vec{r}_1$$

$$\vec{s} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

Let us imagine our frame of reference is another object, which is travelling along the object we were discussing in earlier case.

Then position vector for that object from the new frame of reference is always same. therefore, displacement vector is zero for the object in the case. This can be simplified by relative motion.



$$\vec{r} = \vec{r} + \vec{r}' \quad , \quad \vec{r}' = \vec{r} - \vec{r} \rightarrow \text{vectorial example of position vectors in two frames of reference.}$$

Motion in 1-Dimension

Motion in 1-Dimension is often called Motion in straight line, here all position vectors have only one component $\hat{i}$.

Velocity $\Rightarrow$ velocity is defined by change in position in a unit time (s).

$$\text{avg Velocity} \Rightarrow \bar{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

x is position at a given time.

t_2 is final time & t_1 initial.

$$\text{Instantaneous Velocity} \Rightarrow v = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} = \dot{x}$$

acceleration $\Rightarrow$ change in velocity in a unit time.

$$a = \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}$$

$$a = \frac{dv}{dt}$$

$$a = \dot{v}$$

$$a = \ddot{x}$$

$$a = \frac{d^2x}{dt^2}$$

* Note $\Rightarrow$ Speed is magnitude of velocity vector.

Equations of motion in 1-D

$$v = at + v_0$$

$$r = r_0 + v_0 t + \frac{1}{2} at^2$$

$$v^2 = v_0^2 + 2a(r - r_0)$$

$$\left[\begin{array}{l} v = \text{final velocity} \\ v_0 = \text{initial velocity} \\ r = \text{final position} \\ r_0 = \text{initial position} \\ t = \text{time} \\ a = \text{acceleration} \end{array} \right]$$

Derivatives of vectors in 2-D or 3-D

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

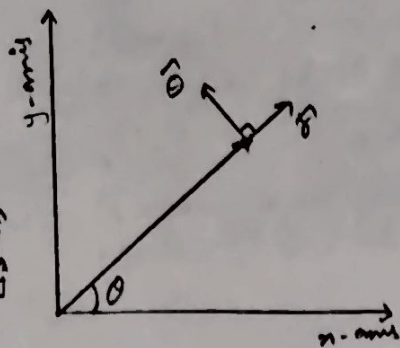
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = \frac{d^2\vec{r}}{dt^2}$$

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

derivatives of individual components.

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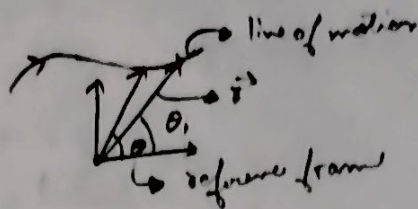
Kinematics in plane polar coordinate system



$$\hat{r} = \cos\theta \hat{i} + \sin\theta \hat{j}$$

$$\hat{\theta} = -\sin\theta \hat{i} + \cos\theta \hat{j}$$

We can imagine motion of a particle in polar coordinates like this,



So our position vector in polar coordinate system is $\vec{r}$.

$$\vec{v} \Rightarrow \frac{d\vec{r}}{dt} \Rightarrow \frac{d(r\hat{r})}{dt} \Rightarrow \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \Rightarrow \hat{r}\frac{dr}{dt} + r\frac{d(\cos\theta\hat{i} + \sin\theta\hat{j})}{dt}$$

product rule

$$\Rightarrow \hat{r}\frac{dr}{dt} + r\left[(-\sin\theta\hat{i} + \cos\theta\hat{j})\cdot\dot{\theta}\right] \Rightarrow \hat{r}\frac{dr}{dt} + r\dot{\theta}\hat{\theta} \Rightarrow v_r\hat{r} + v_\theta\hat{\theta}$$

Velocity vector in polar-coordinates = $\hat{r}\frac{dr}{dt} + r\dot{\theta}\hat{\theta}$ - ***

$$\vec{a} \Rightarrow \frac{d\vec{v}}{dt} \Rightarrow \frac{d(\hat{r}\frac{dr}{dt})}{dt} + \frac{d(r\dot{\theta}\hat{\theta})}{dt}$$

$$\Rightarrow \frac{d\hat{r}}{dt}\frac{dr}{dt} + \frac{dr}{dt}\dot{\hat{r}} + \frac{dr}{dt}\dot{\theta}\hat{\theta} + \frac{d\dot{\theta}}{dt}r\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$\Rightarrow \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + \ddot{\theta}\hat{\theta}r + r\dot{\theta}(\dot{\theta}\hat{x} - \hat{r})$$

$$\Rightarrow \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$

Acceleration Vector $\vec{a} \Rightarrow (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$ - ***

Thus we have obtained an expression for velocity in terms of its radial & angular components.

+ Acceleration

In polar coordinates system motion can be defined in three ways :-

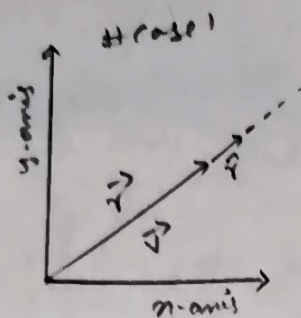
$\vec{r}$ is constant $\rightarrow \dot{r} = 0$

$\hat{\theta}$ is constant $\rightarrow \dot{\theta} = 0$

None of them are constant $\rightarrow \dot{r} \neq 0, \dot{\theta} \neq 0$

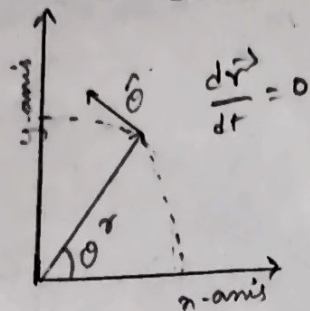
Ex:-

$$\frac{d\hat{\theta}}{dt} = 0$$



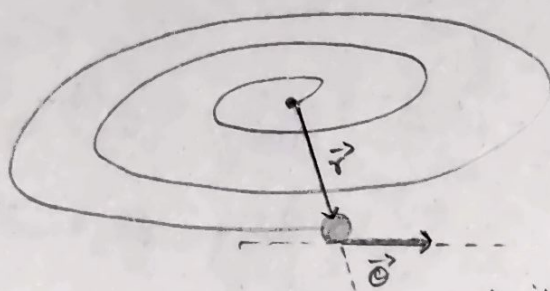
A ball was thrown in an angle θ

case 2



A ball is rotated using a thread in a circular motion

case 3



$$\frac{d\vec{r}}{dt} \neq 0$$

$$\frac{d\hat{\theta}}{dt} \neq 0$$

rotational motion of ball which is rotated using thread and constantly thread is elongated along outside.

or

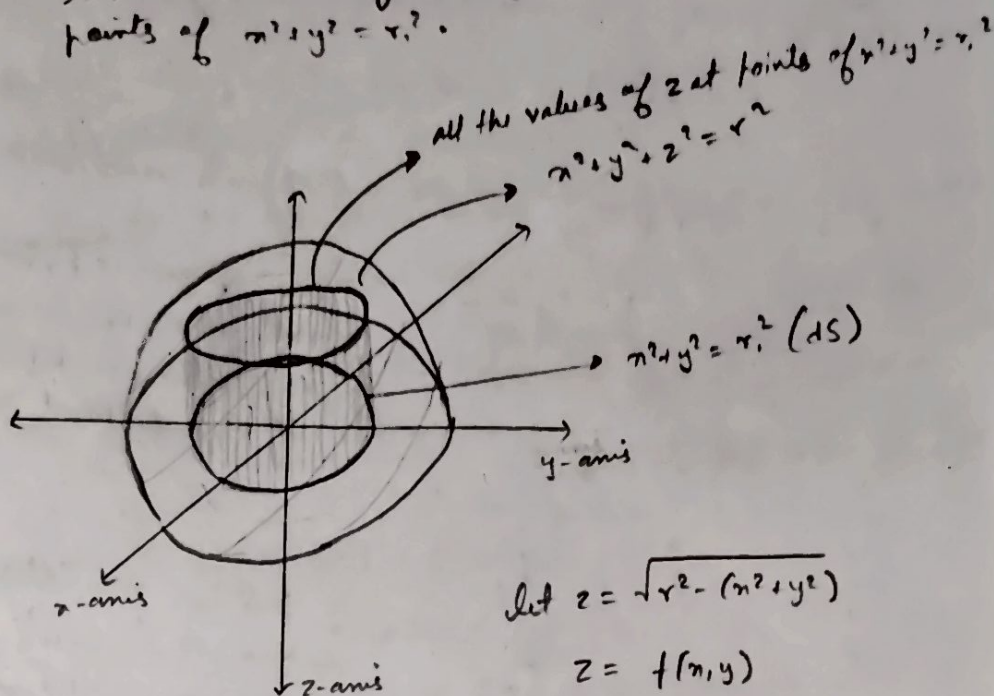
The classical example of bead in a cycle ring question.

[Signature]

Line Integral

Let us imagine a 3D space where we have z as $z = \sqrt{r^2 - (x^2 + y^2)}$ (Hemisphere) and we want to integrate values of z for a specified path let us assume it as $x^2 + y^2 = r^2$ ($r_1 < r$). This can be done by line integral.

It means we are integrating function $x^2 + y^2, z^2 = r^2$ in points of $x^2 + y^2 = r^2$.



$$\text{let } z = \sqrt{r^2 - (x^2 + y^2)}$$

$$z = f(x, y)$$

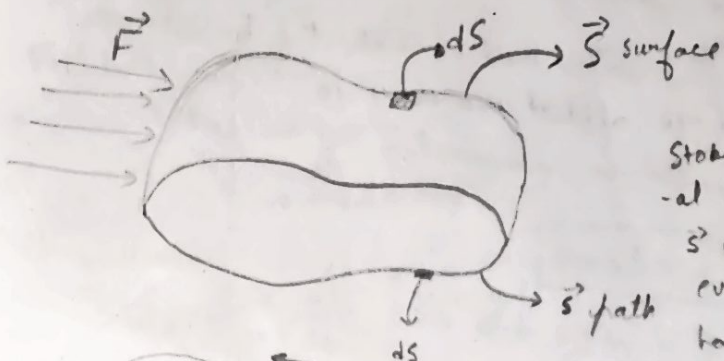
if $x^2 + y^2 = r^2$ be a path defined by dS as smallest movement in the path.

$$\text{line Integral} = \oint f(x, y) \cdot dS$$

$$= \oint f(r \cos \theta, r \sin \theta) \cdot r \cdot d\theta$$

$$= \oint f(r \cos \theta, r \sin \theta) \cdot r \cdot d\theta \text{ --- line integral}$$

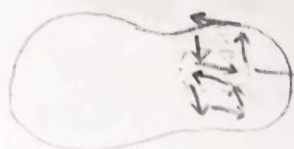
Stoke's Theorem



$$\oint \vec{F} \cdot d\vec{S} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

Stoke's Theorem states that line Integral of a vector field across a path S^1 is equal to Integral of curl of every point on the surface which has boundary same as or path as S^1 in this case S^1 .

$\Rightarrow$



This will remove each other and final curl at every point sum will give 0 if magnitude will be 0.

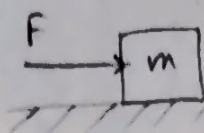
Newton's Laws of Motion

1st:- An object at rest or moving with uniform velocity continues to be at rest or move with uniform velocity until an external force is applied.

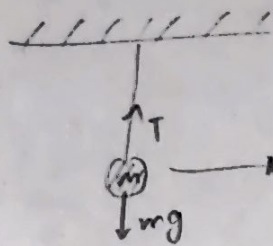
2nd:- Force is equivalent to rate of change of momentum.

$$F = \frac{d(mv)}{dt} = m \frac{dv}{dt} + v \frac{dm}{dt} \quad [m = \text{function of time \& } v \text{ is also function of } t]$$

3rd:- Every force has an equal & opposite reaction.

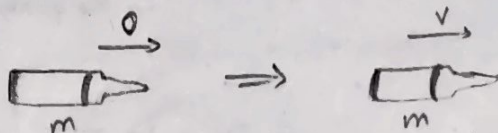


Imagine a person is pushing a block, equal force will be applied to person as pressure on his hands which will try to bend his/hor hands.



In this case gravitational force is acting on the ball & equal tension is acting on that gravity.

Change in momentum:-



In this case let us assume there was a bullet in gun which was in the magazine and had zero velocity, so momentum at that point will be $m \times 0 = 0$, and after few minutes the bullet was shot & it gained velocity v then momentum will be mv .

So Newton's 2nd law states that to make the bullet move there must be a force applied according to 1st law. And that force is equal to change in momentum of the bullet.

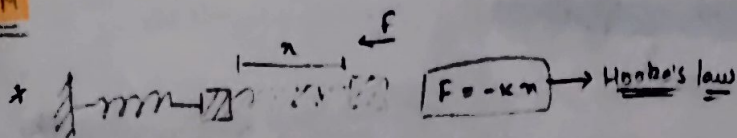
$$\vec{F} = -\vec{\nabla} V$$

$$-\oint \vec{F} \cdot d\vec{r} = V$$

$$F = \Delta mv$$

$$F = m \Delta v \rightarrow \Delta v = \frac{dv}{dt} = a$$

SHM



SHM there is a beautiful relation that this spring oscillation is nothing but motion of projection of particles motion in circular motion

Spring Motion

ω — angular velocity
 $v = \omega x$ — velocity of the object
 $x = A \cos(\omega t)$ — displacement
 $U_0 = \frac{1}{2} k x_0^2 = \frac{1}{2} m v_0^2$ — K.E.
 $\frac{dx}{dt} = -A \sin(\omega t) \omega$
 $\frac{d^2x}{dt^2} = -A \omega^2 \cos(\omega t)$
 $a = -\omega^2 x$

$$\therefore \frac{d^2x}{dt^2} + \omega^2 x = 0 \rightarrow **$$

+

$$m \frac{d^2x}{dt^2} + kx = 0 \rightarrow **$$

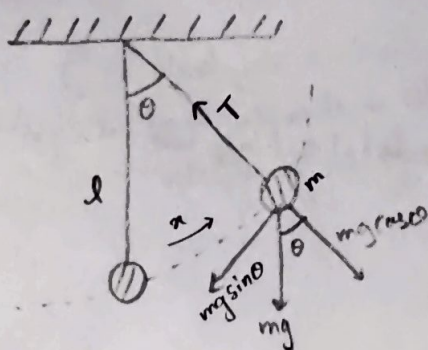
Hooke's law

$$\omega_0 = \sqrt{\frac{k}{m}} \rightarrow **$$

$$\omega_0 = 2\pi f \Rightarrow T = \frac{2\pi}{\omega_0} = \frac{2\pi}{f} \rightarrow **$$

Simple Pendulum

$$x = B \sin(\omega_0 t) + C \cos(\omega_0 t) \rightarrow **$$



Torque in the case of pendulum:-

$$I \alpha \Rightarrow m l^2 \alpha$$

$$l \times m g \sin \theta$$

$$m l^2 \alpha = -m g l \sin \theta$$

$$\alpha = -\frac{g}{l} \sin \theta$$

$$\alpha = -\frac{g}{l} \sin \theta \rightarrow **$$

$$\alpha = -\frac{g}{l} \theta \rightarrow **$$

This motion is also related to motion of spring oscillator

$$\Rightarrow \alpha = -\omega^2 \theta$$

$$\therefore -\omega^2 \theta = -\frac{g}{l} \theta$$

$$\omega = \sqrt{\frac{g}{l}} \rightarrow **$$

Other ways to describe this motion:-

$$m \frac{d^2x}{dt^2} = -kx \quad \left[k = \frac{mg}{l} \right]$$

$$T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi \sqrt{\frac{l}{g}} \rightarrow **$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{l}} \rightarrow ** \quad x = A \cos(\omega t + \phi)$$

Conservation of Energy

Since the force acting in SHM is conservative the total Energy will be constant or we can say conserved.

$$\vec{\nabla} \cdot \vec{F} = 0 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kx & 0 & 0 \end{vmatrix} = 0 \rightarrow \text{force is conservative.}$$

x is given as $x = A \cos(\omega t + \phi)$

$$K.E = \frac{1}{2} m v^2$$

$$K.E = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2$$

$$= \frac{1}{2} m \left(-A\omega \sin(\omega t + \phi) \right)^2$$

$$K.E = \frac{1}{2} m A^2 \omega^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} m A^2 \left(\sqrt{\frac{k}{m}} \right)^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} k A^2 \sin^2(\omega t + \phi)$$

$$P.E = - \int_0^x F \cdot dx$$

$$= - \frac{kx^2}{2}$$

$$= \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

* In case if force is conservative total Energy in the system should be constant.

$\Rightarrow$

$$T.E = K.E + P.E$$

$$= \frac{1}{2} k A^2 (\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi))$$

$$\boxed{T.E = \frac{1}{2} k A^2} \text{ — Hence Proved}$$

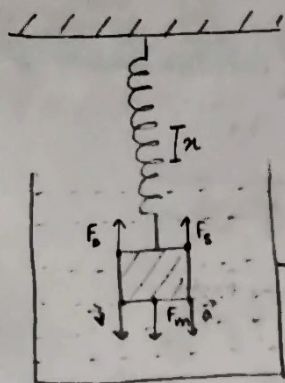
Damped Harmonic motion

In any oscillation we have studied so far it is assumed that there is energy loss in the system, oscillation will keep continuing till infinite time, but there is no such case in real world example. There is any kind of force which will stop this motion. (damping)

Adf

It might be air resistance, friction of surface on which the body is moving etc.

damped harmonic motion tells us the case where that external force is due to air resistance.



Let us imagine that there is a fluid around the block suspended to spring with spring constant 'k'. That fluid opposes the motion of that block when it is pulled a distance downwards. Since it was in equilibrium in standing we can ignore acceleration due to gravity.

By Applying Newton's law of motion, we can say that Net force F_m is combination or sum of all acting forces.

$$\therefore \vec{F}_m = \vec{F}_o + \vec{F}_s$$

$\vec{F}_o \Rightarrow$ Fluid has a viscous property, that means it will oppose motion by applying force opposite to the motion. which will be proportional to the velocity of the object having motion.

$$\vec{F}_o \propto \vec{v}$$

$$\vec{F}_o = b\vec{v}$$

$$\Rightarrow \vec{F}_o = b \frac{dn}{dt} \quad \text{--- **}$$

$\vec{F}_s \Rightarrow$ Spring will try to recoil as the motion is stretching the string by x distance. It will apply force opposite to the displacement x .

$$\vec{F}_s = -kx \quad \rightarrow \quad \underline{\text{Hooke's Law}}$$

$\vec{F}_m \Rightarrow$ Since force applied is total sum of all acting force F_m is sum of F_o + F_s .

$$F_m = -b\vec{v} - kx$$

$$\Rightarrow ma = -b \frac{dn}{dt} - kx$$

$$\Rightarrow m \frac{d^2x}{dt^2} = -b \frac{dn}{dt} - kx$$

[a is acceleration of object]

[a = $\frac{d^2x}{dt^2}$ as x is changing]

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$$\Rightarrow m \frac{d^2 n}{dt^2} + b \frac{dn}{dt} + kn = 0 \rightarrow \text{Second order differential Equation in terms of } n.$$

$$\Rightarrow \frac{dn}{dt^2} + \left[\frac{b}{m} \right] \frac{dn}{dt} + \left[\frac{k}{m} \right] n = 0$$

$\rightarrow \gamma = \text{damping constant}$
 $\hookrightarrow \omega_0^2$ [This projection of circular motion with ω_0]

$$\Rightarrow \frac{d^2 n}{dt^2} + \gamma \frac{dn}{dt} + \omega_0^2 n = 0 \rightarrow \text{Solve this}$$

$$\boxed{\gamma = \frac{b}{m}}$$

$$\boxed{\gamma = \frac{\omega_0}{Q}}$$

$$n = A e^{-\gamma t/2} \cos(\omega_1 t + \phi) \quad \text{--- **}$$

whereas this part shows the decreasing Amplitude or position n .

This part of n represents motion of oscillation

Note:- $\omega_1 \neq \omega_0$ which is just to represent projection of that circular motion.

Since damping decreases frequency or angular velocity we consider in overall motion. that will be reduced by this eqⁿ.

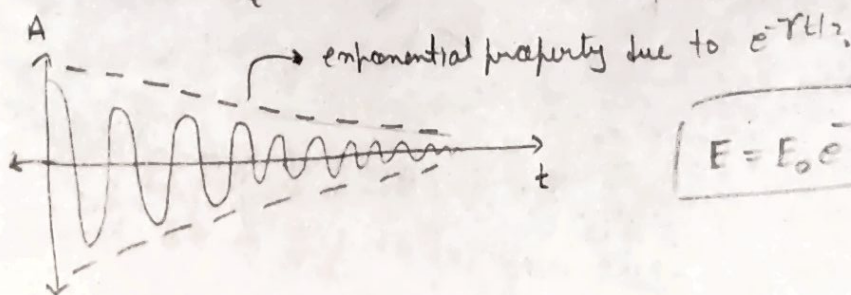
$$\Rightarrow \omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \rightarrow \text{decrease due to damping}$$

$\downarrow$
 Overall angular velocity of oscillation.

$$\boxed{\omega_0^2 \geq \frac{\gamma^2}{4}} \quad \text{--- **}$$

Amplitude of the damped motion will reduce exponentially i.e.

$$A_t = A e^{-\gamma t/2} \rightarrow \text{Just to imagine **}$$



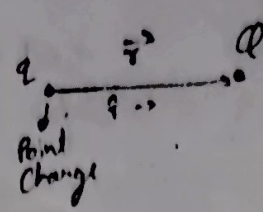
$$\boxed{E = E_0 e^{-\gamma t}} \quad \text{--- **}$$

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Note:- * work done by conservative force in closed path is zero.
 If work done is non zero then it is non-conservative.

Electrostatics

Coulomb's law

$$F = \frac{1}{4\pi\epsilon_0} \times \frac{qQ}{r^2} \hat{r}$$


Point Charge

$$\Rightarrow \epsilon_0 = 8.85 \times 10^{-12}$$

$$\frac{1}{4\pi\epsilon_0} = k = 8.99 \times 10^9 \text{ Nm}^2/\text{C}^2$$

This Eqn is for calculating Electrostatic force between two point charges in rest position.

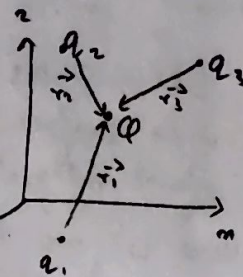
Superposition theorem

$$F = F_1 + F_2 + \dots + F_n$$

force due to Every charge in surrounding

$$\Rightarrow \frac{1}{4\pi\epsilon_0} Q \left(\frac{q_1}{r_1^2} \hat{r}_1 + \frac{q_2}{r_2^2} \hat{r}_2 + \frac{q_3}{r_3^2} \hat{r}_3 + \dots \right)$$

$$\text{Similarly } \Rightarrow F(r) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i^2} \hat{r}_i$$



Superposition theorem says that summation of every force acting on a point charge is equal to the total force applied on the charge.

Electric fields due to continuous charge distributions

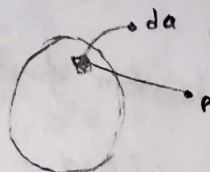
This not necessary that electric field is produced due to only point charge it could be continuous charge, surface, line or volume charges.



Line charge
 $\Rightarrow dq = \lambda dl$



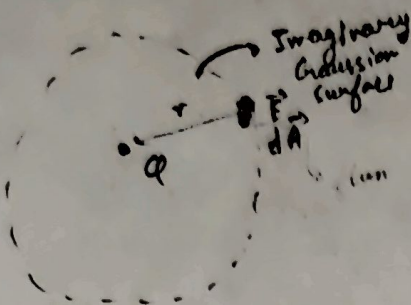
Surface charge
 $\Rightarrow dq = \sigma da$



Volume charge
 $\Rightarrow dq = \rho dv$

by using these small charge elements and theorem we can find electric field and other stuff.

Gauss Law



Total Electric flux in the Gaussian surface is equal to

$$\Rightarrow \oint_S \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q}{\epsilon_0} - ***$$

$(d\vec{A} = r^2 \sin\theta d\theta d\phi \hat{r})$ for polar coordinates or spherical surface.

Electric flux

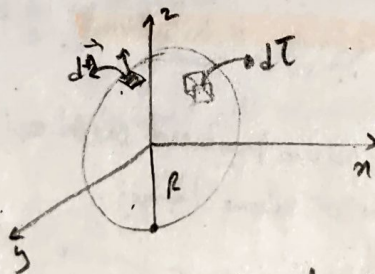
$$\Rightarrow \int_S \vec{E} \cdot d\vec{A} \text{ or } E \cdot A \text{ or } EA \cos\theta$$

Gauss's Divergence theorem

Like Stoke's theorem where line integral can be converted into surface integral, we can convert surface integral to volume integral.

Here we are doing it in Gauss's law

$$\Rightarrow \oint_S \vec{E} \cdot d\vec{A} = \int_V (\nabla \cdot \vec{E}) dV$$

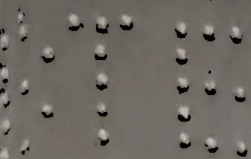


rewriting Gauss's law for specific volume

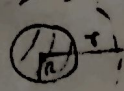
$$\Rightarrow Q = \int_V \rho dV$$

$$\Rightarrow \int_V (\nabla \cdot \vec{E}) dV = \int_V \frac{\rho}{\epsilon_0} dV \Rightarrow \boxed{\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$$

Gauss's Surface

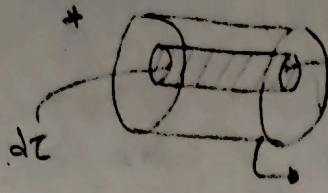


*



spherical Gauss surface for spherical charge

*

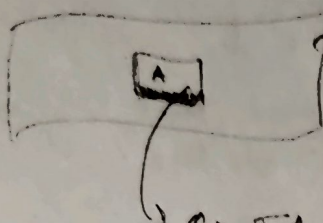


cylindrical charge

cylindrical Gaussian surface

* Gauss's Pillbox

$\Rightarrow \int \rho d\tau = q_{enc} \rightarrow$ Integrate volume charge



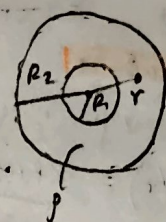
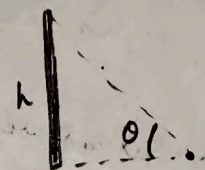
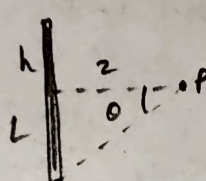
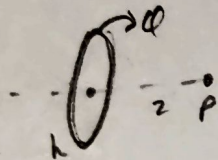
Surface with charge spread across all over

$$Q = \sigma A$$

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\epsilon_0}$$

$$V = -\int \mathbf{E} \cdot d\mathbf{r} \quad \& \quad \mathbf{E} = -\nabla V$$

Using this we can find electric field for :-



And many more.

Poisson's + Laplacian Equation

* $\mathbf{E} = -\nabla V$ Electric field is equal to Gradient of scalar unit potential.

Divergence of Electric field

$$\Rightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

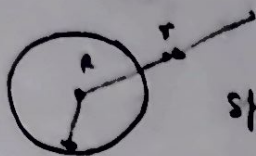
$$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V \quad \left(\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right)$$

$$\Rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}} \text{ — Poisson's Eqn}$$

000

Notes:- $\vec{d\vec{l}}$ for polar coordinates
 $= dr\hat{r} + r d\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$

Imp



spherical shell potential inside & outside

Outside $\Rightarrow E \cdot d\vec{l} = \frac{kq}{r^2} dr \Rightarrow \boxed{-\frac{kq}{r}}$

$\therefore V(r) = -\left(-\frac{kq}{r}\right) = \frac{kq}{r} \quad \text{--- (1)}$

Inside $\Rightarrow$

$$V(r) = -\int_{\infty}^R \frac{kq}{(r')^2} dr' - \int_R^r \frac{kq}{(r')^2} dr'$$

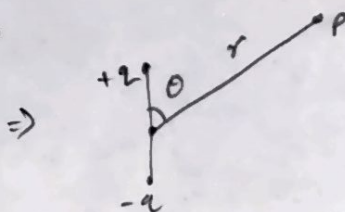
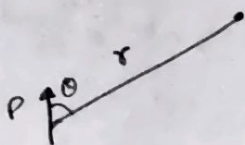
$$= \frac{kq}{r'} \Big|_{\infty}^R + 0$$

$V(r) = \frac{kq}{r} \quad \text{--- (2)}$

Electric dipole

$V(r) = \frac{kP \cos\theta}{r^2} \Rightarrow \frac{k \times qd \times \cos\theta}{r^2} \Rightarrow \frac{1}{4\pi\epsilon_0} \times \frac{2d \cos\theta}{r^2}$ (Learn by heart)

$E(r, \theta) = \frac{kP}{r^3} (2\cos\theta\hat{r} + \sin\theta\hat{\theta})$



(diagrams)

Imp

Dielectrics

* Assume a material which has natural dipole moment due to its Atomic structure, it will be dielectric & its polarisation can be defined by dipole moment per unit volume.

$$P \equiv \text{dipole moment} / \text{Total volume.}$$

* This type of material will have electric field & potential equivalent to charge

$$\Rightarrow \sigma_b = P \cdot \hat{n} \quad (\text{Surface charge density}) \quad - (1)$$

$$P_b = -\vec{\nabla} \cdot P \quad (\text{Volume charge density}) \quad - (2)$$

$$[P_b = 0 \text{ if } P \text{ is consistent across material}]$$

Note:- Total charge in a dipole is 0 & polarized objects will also have it zero '0'.

$$Q = \int_{\text{surf}} \sigma_b da + \int_{\text{vol}} P_b dz$$

$$= \int P(r') \cdot \hat{n} da + \int -\vec{\nabla} \cdot P(r) dz$$

$$= \int \vec{\nabla} \cdot P(r) dz - \int \vec{\nabla} \cdot P(r) dz \quad [\text{Gauss theorem}]$$

$$= \boxed{0} - \text{Hence, Proved.}$$

* Questions $\Rightarrow$

$$P(r) = \epsilon_0 \text{ find } \sigma_b \text{ \& } P_b \text{ [use (1) \& (2) to get it]}$$

* In Dielectric there is presence of electric field which is due to
 $\Rightarrow$ bound charge & free charge

$$\Rightarrow P = P_b + P_f$$

$$\Rightarrow \epsilon_0 \vec{\nabla} \cdot \vec{E} = P_b + P_f \Rightarrow \epsilon_0 \vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot \vec{P} + P_f$$

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = P_f \quad - (3)$$

(Electric displacement)

$$\therefore \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

$$\vec{\nabla} \cdot \vec{D} = \rho_f \quad (\vec{D} = \text{electric displacement})$$

$$\Rightarrow \oint \mathbf{D} \cdot d\vec{a} = Q_f \quad (4)$$

total free charge

Linear Dielectrics

Linear dielectrics are dielectrics where induced polarization is proportional to the applied electric field.

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

$\hookrightarrow$ Electric Susceptibility

$$\therefore \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{D} = (1 + \chi_e) \epsilon_0 \mathbf{E}$$

we can say $\epsilon = \epsilon_0 (1 + \chi_e)$ $\xrightarrow{\text{Relative Permittivity}}$

$\hookrightarrow$ Permittivity of material

$$\boxed{\epsilon_r = \frac{\epsilon}{\epsilon_0} = (1 + \chi_e)}$$

Some cases $\Rightarrow \mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{\epsilon_{vac}}{\epsilon_r} \mathbf{D} \quad (\mathbf{D} = \epsilon_0 \mathbf{E}_{vac})$

Dielectrics to Increase Capacitance

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Magnetism

Lorentz force

Lorentz force is the total force applied on a ^{moving} charge with influence of both electric and magnetic field.

$$F_{elec} = qE$$

$$F_{mag} = q(\mathbf{v} \times \mathbf{B})$$

$$\begin{aligned} \text{Total force} \\ \text{or} \\ \text{Lorentz force} \end{aligned} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

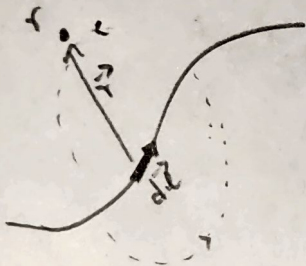
Current in form of density

$\mathbf{J} \rightarrow$ Current volume density

$$\mathbf{J} = \frac{d\mathbf{I}}{da_1} = \rho \mathbf{v}$$

$$\mathbf{F}_{mag} = (\mathbf{v} \times \mathbf{B})q = \int (\mathbf{v} \times \mathbf{B})\rho d\tau = \int (\mathbf{J} \times \mathbf{B})d\tau$$

Biot-Savart law



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{d\mathbf{I} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}$$

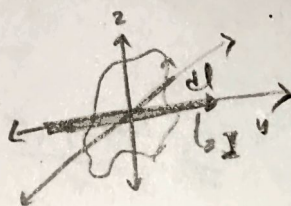
$$[\mu_0 = 4\pi \times 10^{-7}]$$

Divergence and Curl of Magnetic field

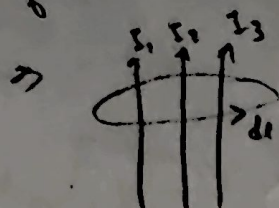
Line integral of Magnetic field is independent of path

$\Rightarrow$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$$



If more than one current



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_1 + \mu_0 I_2 + \mu_0 I_3$$

$$= \mu_0 I_{enc}$$

or

$$= \mu_0 \int \vec{J} \cdot d\vec{a}$$

Note: $\mu_0 \vec{J} = (\vec{\nabla} \times \vec{B})$

So, $\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a}$ $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ — (2)

$$\oint \vec{B} \cdot d\vec{l} = \int \mu_0 \vec{J} \cdot d\vec{a}$$

$$\oint \vec{B} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} \text{ — (1)}$$

Important

(Magnetostatics) $\vec{\nabla} \cdot \vec{B} = 0$	$\vec{\nabla} \times \vec{E} = 0$ (Electrostatics)
$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

Vector Potential

As in Electrostatics $\vec{\nabla} \times \vec{E} = 0 \therefore \vec{E} = -\vec{\nabla} V$

Similarly in Magnetostatics $\vec{\nabla} \cdot \vec{B} = 0$

$\therefore$ We will assume a Vector A

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} = \vec{\nabla} (\vec{\nabla} \times \vec{A}) \\ &= \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} \end{aligned}$$

$$\boxed{-\nabla^2 \vec{A} = \mu_0 \vec{J}} \text{ — **}$$

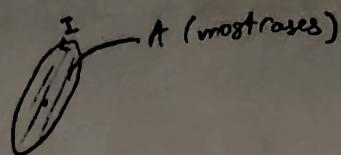
$$A(r) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r') dV'}{r} \text{ or } \frac{\mu_0 I}{4\pi} \int \frac{dl}{r} \text{ (for wire current)}$$

Ans

#

Magnetic dipole

$$m = I \int dA$$



magnetic vector potential
for dipole

$$\Rightarrow A(r) = \frac{\mu_0}{4\pi} \times \frac{m \times \hat{r}}{r^2}$$

$\hat{r}$ pointing towards a point
assume $\hat{m} = \hat{z}$

Using $A(r)$, we can find B

$$\Rightarrow B = \vec{\nabla} \times \vec{A}$$

$$\therefore A(r) = \frac{\mu_0 m \sin \theta}{4\pi r^2} \hat{\phi}$$

$$= -\hat{x} \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 0 & 0 & \frac{\mu_0 m \sin \theta}{4\pi r^2} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \end{vmatrix}$$

$$= \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}) \rightarrow \text{***}$$

Similar to electrostatic

$$\Rightarrow P = P_0 \hat{z} \Rightarrow \vec{E} = \frac{P_0}{4\pi \epsilon_0 r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta})$$

Magnetization

What is magnetization? - magnetic dipole moment per unit volume

* there is some kind of dipole moment caused due to spinning electrons in shells and in their ones.

They can be parallel to $B \rightarrow$ Paramagnet

They can be against to $B \rightarrow$ Diamagnet

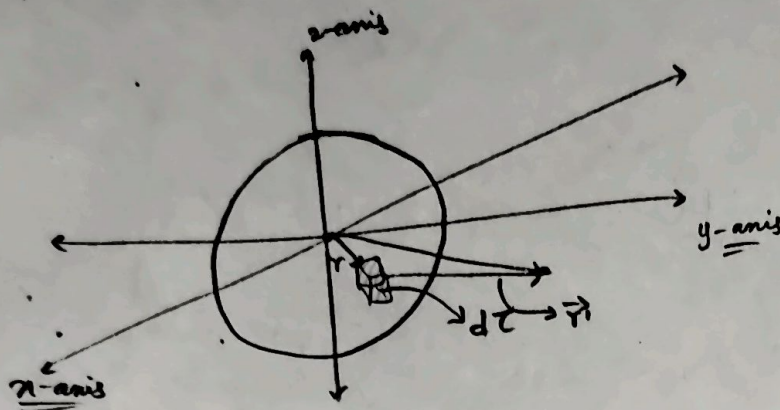
They can be in absence to $B \rightarrow$ Ferromagnet

Magnetic field due to magnetised object

$$\vec{J}_b(\vec{r}) = \vec{\nabla} \times \vec{M}(\vec{r}) \rightarrow \text{Volume current}$$

$$\vec{K}_b(\vec{r}) = \vec{M}(\vec{r}) \times \hat{n} \rightarrow \text{Surface current}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \left(\int \frac{\vec{J}_b(\vec{r}')}{r'} d\tau' + \int \frac{\vec{K}_b(\vec{r}')}{r'} dq' \right)$$



$$\vec{J}_b = \text{Volume Current} \Rightarrow \vec{\nabla} \times \vec{M}(\vec{r})$$

$$\text{Surface Current} \Rightarrow \vec{M}(\vec{r}) \times \hat{n}$$

$$\vec{J} = \vec{J}_f + \vec{J}_b$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \Rightarrow \mu_0 (\vec{J}_f + \vec{J}_b) \Rightarrow \mu_0 (\vec{J}_f + \vec{\nabla} \times \vec{M}(\vec{r}))$$

$$\Rightarrow \vec{\nabla} \times \frac{\vec{B}}{\mu_0} - \vec{\nabla} \times \vec{M}(\vec{r}) = \vec{J}_f$$

$$\Rightarrow \vec{J}_f = \vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}(\vec{r}) \right)$$

$\downarrow$
 $\vec{H}$

$$\vec{\nabla} \times \vec{M} = \vec{J}_f \quad \text{--- **}$$

$$\oint \vec{M} \cdot d\vec{\ell} = I_{fenc} \quad \text{--- **}$$

For Linear Media,

$$\vec{M} = \chi_m \vec{H} \quad (\chi_m \text{ is magnetic susceptibility})$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (1 + \chi_m) \vec{H}$$

$\hookrightarrow \mu \rightarrow$ permeability of material

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