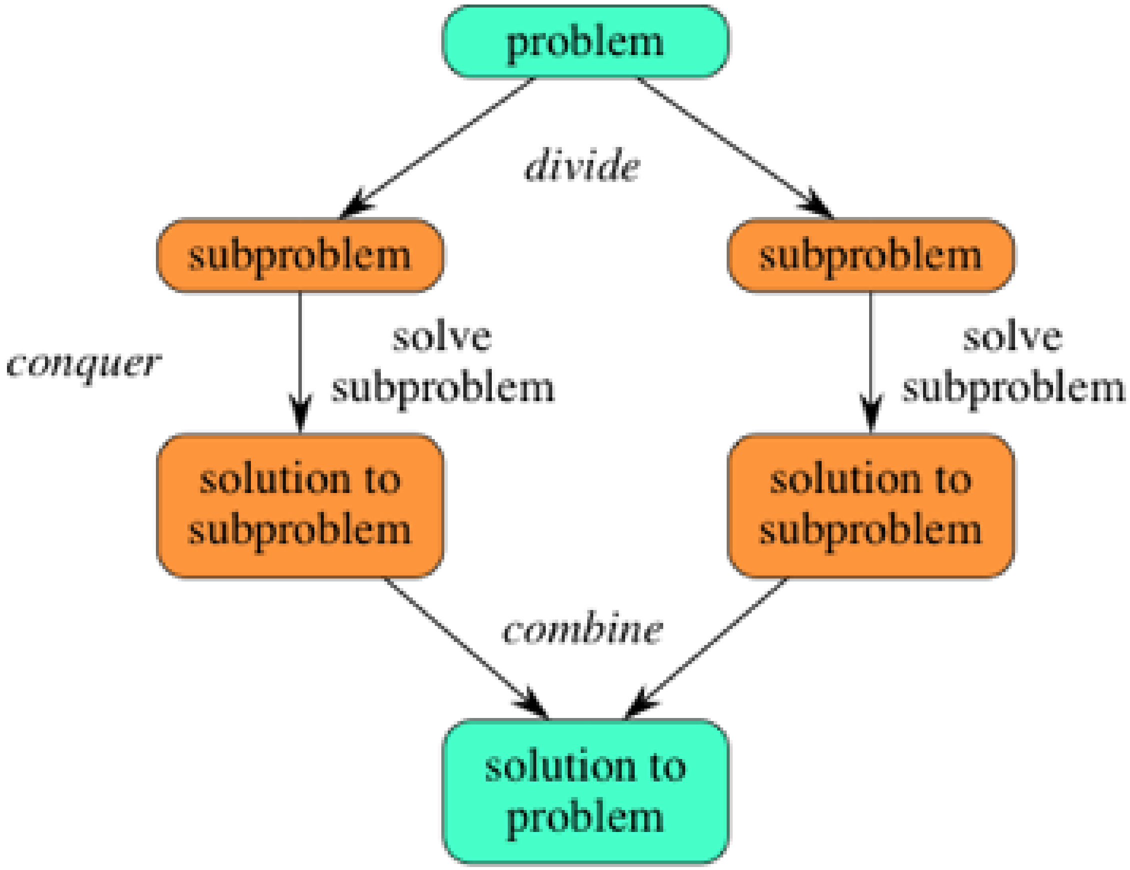
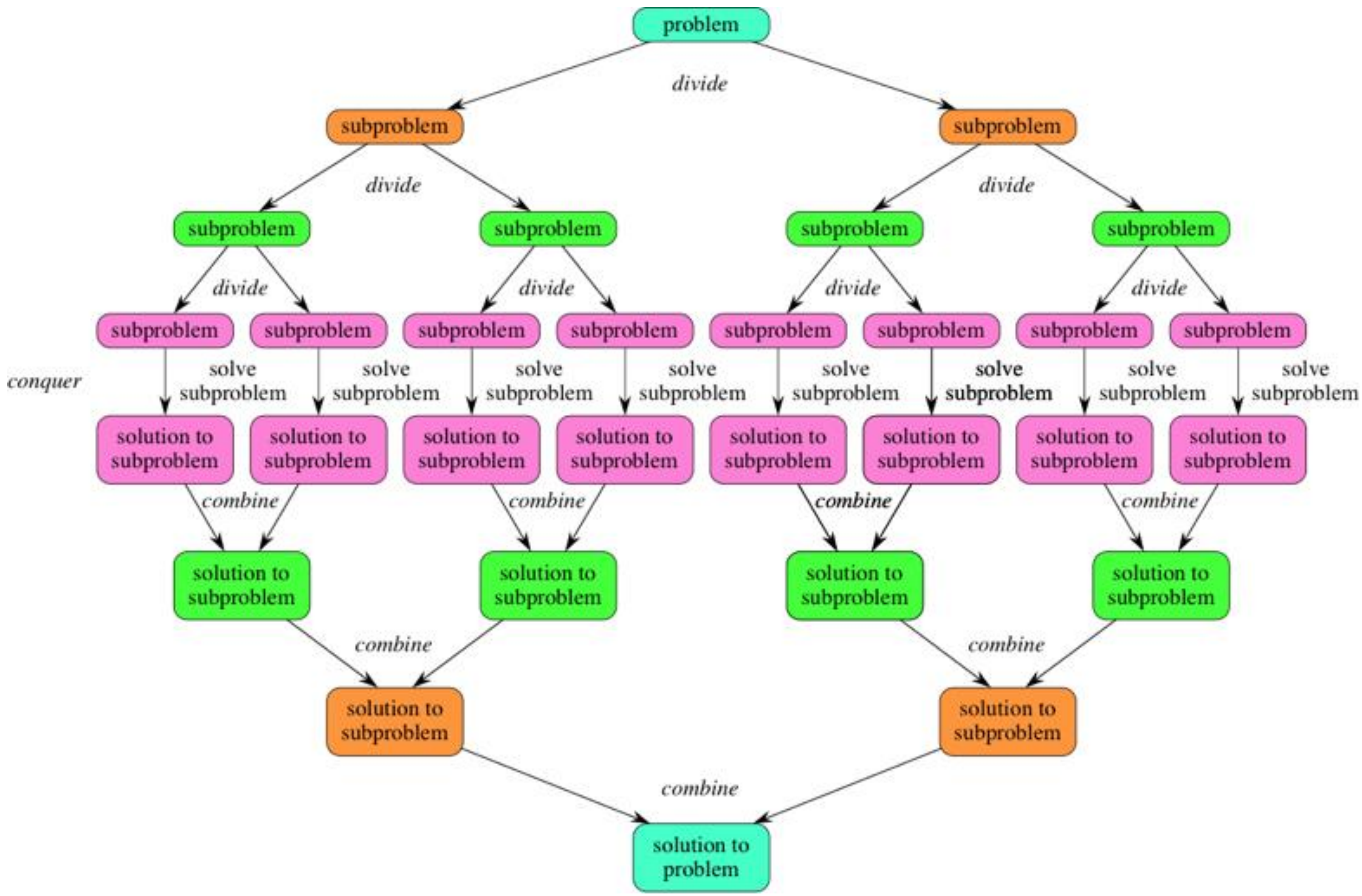
Divide Conquer and Combine Algorithm

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Matrix Multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} (aj + bm + cp) & (ak + bn + cq) & (al + bo + cr) \\ (dj + em + fp) & (dk + en + fq) & (dl + eo + fr) \\ (gj + hm + ip) & (gk + hn + iq) & (gl + ho + ir) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 10 & 11 \\ 20 & 21 \\ 30 & 31 \end{bmatrix}$$

$$= \begin{bmatrix} 1x10 + 2x20 + 3x30 & 1x11 + 2x21 + 3x31 \\ 4x10 + 5x20 + 6x30 & 4x11 + 5x21 + 6x31 \end{bmatrix}$$

$$= \begin{bmatrix} 10+40+90 & 11+42+93 \\ 40+100+180 & 44+105+186 \end{bmatrix} = \begin{bmatrix} 140 & 146 \\ 320 & 335 \end{bmatrix}$$

Matrix Multiplication

SQUARE-MATRIX-MULTIPLY(A, B)

```
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

Cost = $O(n^3)$

Matrix Multiplication

Matrix multiplication. Given two n-by-n matrices A and B, compute $C = AB$.

Grade-school. $\Theta(n^3)$ arithmetic operations.

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\begin{bmatrix} .59 & .32 & .41 \\ .31 & .36 & .25 \\ .45 & .31 & .42 \end{bmatrix} = \begin{bmatrix} .70 & .20 & .10 \\ .30 & .60 & .10 \\ .50 & .10 & .40 \end{bmatrix} \times \begin{bmatrix} .80 & .30 & .50 \\ .10 & .40 & .10 \\ .10 & .30 & .40 \end{bmatrix}$$

Q. Can we perform matrix multiplication more effeciantly

Consider $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 8 & 2 & 2 \\ 5 & 1 & 4 & 9 \\ 6 & 2 & 5 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 9 & 1 & 4 & 5 \\ 2 & 3 & 6 & 7 \end{pmatrix}$. I'll use MATRIX-MULTIPLY-RECURSIVE(MMR) algo to multiply A and B.

Since $n > 1$, we break A, B into eight $\frac{n}{2}$ matrices:

$$A_{11} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix},\ A_{12} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix},\ A_{21} = \begin{pmatrix} 5 & 1 \\ 6 & 2 \end{pmatrix},\ A_{22} = \begin{pmatrix} 4 & 9 \\ 5 & 0 \end{pmatrix}$$
$$B_{11} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},\ B_{12} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix},\ B_{21} = \begin{pmatrix} 9 & 1 \\ 2 & 3 \end{pmatrix},\ B_{22} = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$$

Now $\mathrm{MMR}(A_{11}, B_{11}, C_{11}, 2)$ produces eight matrices which are integers. I'll use red indices to distinguish these matrices from the forgoing ones:

$$A_{\color{red}{11}} = 1, A_{\color{red}{12}} = 2, A_{\color{red}{21}} = 3, A_{\color{red}{22}} = 8$$
$$B_{\color{red}{11}} = 1, B_{\color{red}{12}} = 2, B_{\color{red}{21}} = 3, B_{\color{red}{22}} = 4$$

Applying the matrix-multiplication procedure on the integers above, we have

$$C_{\color{red}{11}} = 1 \cdot 1 + 2 \cdot 3 = 7$$
$$C_{\color{red}{12}} = 1 \cdot 2 + 2 \cdot 4 = 10$$
$$C_{\color{red}{21}} = 3 \cdot 1 + 8 \cdot 3 = 24$$
$$C_{\color{red}{22}} = 3 \cdot 2 + 8 \cdot 4 = 38$$

which give us $C_{11} = \mathrm{MMR}(A_{12}, B_{21}, C_{11}, 2) = \begin{pmatrix} 13 & 7 \\ 22 & 8 \end{pmatrix}$.

Applying the matrix multiplication rule again, the sum

$$\begin{aligned} &\mathrm{MMR}(A_{11}, B_{11}, C_{11}, 2) + \mathrm{MMR}(A_{12}, B_{21}, C_{11}, 2) \\ &= \begin{pmatrix} 7 & 10 \\ 24 & 38 \end{pmatrix} + \begin{pmatrix} 13 & 7 \\ 22 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 20 & 17 \\ 46 & 46 \end{pmatrix} \end{aligned}$$

fills in the first quarter of the resultant matrix $A \cdot B = C = \begin{pmatrix} 20 & 17 & x & x \\ 46 & 46 & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix}$. We fill in the rest of C in a similar manner.

```
MATRIX-MULTIPLY-RECURSIVE(A, B, C, n)
  if n == 1
  // Base case.
    c_11 = c_11 + a_11 · b_11
    return
  // Divide.
  partition A, B, and C into n/2 × n/2 submatrices A_11,
A_12, A_21, A_22; B_11, B_12, B_21, B_22;
     and C_11, C_12, C_21, C_22; respectively
  // Conquer.
  MATRIX-MULTIPLY-RECURSIVE(A_11, B_11, C_11, n/2)
  MATRIX-MULTIPLY-RECURSIVE(A_11, B_12, C_12, n/2)
  MATRIX-MULTIPLY-RECURSIVE(A_21, B_11, C_21, n/2)
  MATRIX-MULTIPLY-RECURSIVE(A_21, B_12, C_22, n/2)
  MATRIX-MULTIPLY-RECURSIVE(A_12, B_21, C_11, n/2)
  MATRIX-MULTIPLY-RECURSIVE(A_12, B_22, C_12, n/2)
  MATRIX-MULTIPLY-RECURSIVE(A_22, B_21, C_21, n/2)
  MATRIX-MULTIPLY-RECURSIVE(A_22, B_22, C_22, n/2)
```

Block Matrix Multiplication

$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$

Matrix Multiplication: Warmup

To multiply two n-by-n matrices A and B:

- Divide: partition A and B into $\frac{1}{2}n$-by-$\frac{1}{2}n$ blocks.
- Conquer: multiply 8 pairs of $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices, recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

$$T(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow T(n) = \Theta(n^3)$$

Divide Conquer and Combine Algorithm

Strassen Matrix multiplication: Fast way to multiply 2 matrices

Fast Matrix Multiplication

Key idea. multiply 2-by-2 blocks with only 7 multiplications.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} P_1 &= A_{11} \times (B_{12} - B_{22}) \\ P_2 &= (A_{11} + A_{12}) \times B_{22} \\ P_3 &= (A_{21} + A_{22}) \times B_{11} \\ P_4 &= A_{22} \times (B_{21} - B_{11}) \\ P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

- 7 multiplications.
- $18 = 8 + 10$ additions and subtractions.

Fast Matrix Multiplication

To multiply two n-by-n matrices A and B: [Strassen 1969]

- Divide: partition A and B into $\frac{1}{2}n$-by-$\frac{1}{2}n$ blocks.
- Compute: 14 $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices via 10 matrix additions.
- Conquer: multiply 7 pairs of $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices, recursively.
- Combine: 7 products into 4 terms using 8 matrix additions.

Analysis.

- Assume n is a power of 2.
- $T(n)$ = # arithmetic operations.

$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

Thank You