Linear Discriminant Analysis (LDA)

Lecture 17

Dimensionality Reduction Techniques (LDA)

- **Linear Discriminant Analysis (LDA) :**
- Linear Discriminant Analysis (LDA) is used to **solve dimensionality reduction** for data with higher attributes.
- **History :** The original dichotomous discriminant analysis was developed by Sir Ronald Fisher in **1936**.
- **Introduction :**
- Pre-processing step for pattern-classification and machine learning applications.
- Used for feature extraction.
- Linear transformation that maximize the separation between multiple classes.

ML | Linear Discriminant Analysis

- It is a supervised learning algorithm used for classification tasks in machine learning. It is a technique used to find a linear combination of features that best separates the classes in a dataset.

- LDA works by projecting the data onto a lower-dimensional space that maximizes the separation between the classes. It does this by finding a set of linear discriminants that maximize the ratio of between-class variance to within-class variance. In other words, it finds the directions in the feature space that best separates the different classes of data.

- LDA assumes that the data has a Gaussian distribution and that the covariance matrices of the different classes are equal. It also assumes that the data is linearly separable, meaning that a linear decision boundary can accurately classify the different classes.

LDA has several advantages, including:

- It is a simple and computationally efficient algorithm.
  It can work well even when the number of features is much larger than the number of training samples.
  It can handle multicollinearity (correlation between features) in the data.

However, LDA also has some limitations, including:

- It assumes that the data has a Gaussian distribution, which may not always be the case.
  It assumes that the covariance matrices of the different classes are equal, which may not be true in some datasets.
  It assumes that the data is linearly separable, which may not be the case for some datasets.
  It may not perform well in high-dimensional feature spaces.

Difference between Linear Discriminant Analysis and PCA

- **PCA is an unsupervised algorithm** that does not care about classes and labels and only aims to find the principal components to maximize the variance in the given dataset. At the same time, **LDA is a supervised algorithm** that aims to find the linear discriminants to represent the axes that maximize separation between different classes of data.
- **LDA is much more suitable for multi-class classification tasks compared to PCA**. However, PCA is assumed to be an as good performer for a comparatively small sample size.
- Both LDA and PCA are used as dimensionality reduction techniques, where PCA is first followed by LDA

Goal

To **classify observations** into 2 or more groups based on k discriminant functions

(Dependent variable Y is categorical with k classes.)

Assumptions

Multivariate Normal Distribution

- variables are distributed normally within the classes/groups.

Similar Group Covariances

- Correlations between and the variances within each group should be similar.

Dependent Variable

Must be categorical with 2 or more classes (groups).

If there are only 2 classes, the discriminant analysis procedure will give the same result as the multiple regression procedure.

Independent Variables

Continuous or categorical independent variables

If categorical, they are converted into binary (dummy) variables as in multiple linear regression

Example 1

🧠 Problem Statement

You have the following dataset with two classes (A and B), each with 2 data points and 2 features (x_1 and x_2):

Class A:

- Point 1: (2, 3)
- Point 2: (3, 3)

Class B:

- Point 3: (6, 5)
- Point 4: (7, 8)

✅ Step 1: Compute Class Means

$$\mu_A = \frac{1}{2}[(2,3) + (3,3)] = (2.5, 3.0)$$

$$\mu_B = \frac{1}{2}[(6,5) + (7,8)] = (6.5, 6.5)$$

✅ Step 2: Compute the Within-Class Scatter Matrix (S_W)

$$S_W = S_A + S_B$$

For Class A:

$$S_A = \sum(x - \mu_A)(x - \mu_A)^T = [(2,3) - (2.5,3)]^T[(2,3) - (2.5,3)] + [(3,3) - (2.5,3)]^T[(3,3) - (2.5,3)]$$

$$= \begin{bmatrix} -0.5 \\ 0 \end{bmatrix} \begin{bmatrix} -0.5 & 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}$$

For Class B:

$$S_B = [(6,5) - (6.5,6.5)]^T[(6,5) - (6.5,6.5)] + [(7,8) - (6.5,6.5)]^T[(7,8) - (6.5,6.5)]$$

$$= \begin{bmatrix} -0.5 \\ -1.5 \end{bmatrix} \begin{bmatrix} -0.5 & -1.5 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} \begin{bmatrix} 0.5 & 1.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25 & 0.75 \\ 0.75 & 2.25 \end{bmatrix} + \begin{bmatrix} 0.25 & 0.75 \\ 0.75 & 2.25 \end{bmatrix} = \begin{bmatrix} 0.5 & 1.5 \\ 1.5 & 4.5 \end{bmatrix}$$

$$S_W = S_A + S_B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.5 & 1.5 \\ 1.5 & 4.5 \end{bmatrix} = \begin{bmatrix} 1.0 & 1.5 \\ 1.5 & 4.5 \end{bmatrix}$$

✅ Step 3: Compute the Between-Class Scatter Matrix

$$S_B = (\mu_A - \mu_B)(\mu_A - \mu_B)^T$$

$$= \begin{bmatrix} -4.0 \\ -3.5 \end{bmatrix} \begin{bmatrix} -4.0 & -3.5 \end{bmatrix} = \begin{bmatrix} 16 & 14 \\ 14 & 12.25 \end{bmatrix}$$

Step 4: Final Step

✅ LDA Eigenvalue Equation

$$S_W^{-1} S_B \, v = \lambda v$$

Where:

- S_W: Within-class scatter matrix
- S_B: Between-class scatter matrix
- v: Eigenvector (projection direction)
- λ: Eigenvalue

We already have the matrices from the earlier steps:

✅ Step: Compute $S_W^{-1} S_B$

We already calculated S_W^{-1} in the last message:

$$S_W^{-1} = \frac{1}{2.25} \begin{bmatrix} 4.5 & -1.5 \\ -1.5 & 1.0 \end{bmatrix} = \begin{bmatrix} 2.0 & -0.6667 \\ -0.6667 & 0.4444 \end{bmatrix}$$

Now compute:

$$S_W^{-1} S_B = \begin{bmatrix} 2.0 & -0.6667 \\ -0.6667 & 0.4444 \end{bmatrix} \cdot \begin{bmatrix} 16 & 14 \\ 14 & 12.25 \end{bmatrix}$$

Let's compute each element:

$$S_W^{-1} S_B = \begin{bmatrix} 22.6662 & 19.8334 \\ -4.4456 & -3.8894 \end{bmatrix}$$

🔷 First row:

- [1,1]: $(2.0)(16) + (-0.6667)(14) = 32 - 9.3338 = 22.6662$
- [1,2]: $(2.0)(14) + (-0.6667)(12.25) = 28 - 8.1666 = 19.8334$

🔷 Second row:

- [2,1]: $(-0.6667)(16) + (0.4444)(14) = -10.6672 + 6.2216 = -4.4456$
- [2,2]: $(-0.6667)(14) + (0.4444)(12.25) =$ [illegible]$.3338 + 5.4444 = -3.8894$

✅ Step: Solve Eigenvalue Problem

Now, to find the LDA projection vector v, solve the eigenvalue problem:

$$S_W^{-1} S_B \cdot v = \lambda v$$

This is a **standard eigenvalue problem**. Solving it gives:

- The **eigenvector** v = direction of maximum class separability
- The **eigenvalue** λ = amount of separation between classes

We computed the matrix:

$$A = S_W^{-1} S_B = \begin{bmatrix} 22.6662 & 19.8329 \\ -4.4456 & -3.8899 \end{bmatrix}$$

We now want to solve the **eigenvalue equation**:

$$A \cdot v = \lambda v$$

Which can be rearranged as:

$$(A - \lambda I)v = 0$$

To find **λ (eigenvalues)**, we solve:

$$\det(A - \lambda I) = 0$$

🔢 Step-by-Step: Finding Eigenvalues (λ)

Let's denote:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 22.6662 & 19.8329 \\ -4.4456 & -3.8899 \end{bmatrix}$$

The characteristic equation is:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$

Substitute values:

$$(22.6662 - \lambda)(-3.8899 - \lambda) - (19.8329)(-4.4456) = 0$$

You solve this **quadratic equation** in λ to find the eigenvalues.

- $\lambda_1 \approx$ **18.7763** (dominant eigenvalue)
- $\lambda_2 \approx$ **~0** (close to zero, usually ignorable in 2-class LDA)

🔁 Step: Find Eigenvector (v)

Once you know λ, plug it back into:

$$(A - \lambda I)v = 0$$

Let's solve:

$$\left(\begin{bmatrix} 22.6662 & 19.8329 \\ -4.4456 & -3.8899 \end{bmatrix} - 18.7763 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

That simplifies to:

$$\begin{bmatrix} 3.8899 & 19.8329 \\ -4.4456 & 14.8864 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

You can now solve this linear system for the **ratio** between v_1 and v_2. The solution (after normalization) is:

$$v = \begin{bmatrix} 0.9813 \\ -0.1925 \end{bmatrix}$$

This is the **LDA projection direction**, i.e., the vector onto which the data should be projected for **maximum class separability**.

📌 Final Answer:

$$\boxed{w = \begin{bmatrix} 0.9813 \\ -0.1925 \end{bmatrix}}$$

You can now project any 2D point x using $x' = w^T x$ to reduce it to 1D and perform classification.

Let me know if you want to see how to project the original data onto this LDA vector! [>-]

✅ Projection Formula in LDA

$$x' = w^T x$$

Where:

- x is the **original data point** (a column vector or row vector)
- w is the **LDA projection vector** (a column vector)
- x' is the **projected value** (a scalar, 1D point)

🧠 **Example:**

For the point $x = [2, 3]$ and LDA vector $w = [0.9813, -0.1925]$:

$$x' = (0.9813)(2) + (-0.1925)(3) = 1.9626 - 0.5775 = 1.3851$$

Final Projected Dataset

📌 **Projected Values:**

Original Point	Class	Projected Value
(2, 3)	A	1.3851
(3, 3)	A	2.3664
(6, 5)	B	4.9253
(7, 8)	B	5.3291

Example 2

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Q) Factory ABC produces a very expensive and high quality chip rings that their qualities are measured in terms of weight in mg and diameter in cm. Dataset is:

Wt in (mg)	Diameter in (cm)	Result
4	1	Pass
2	4	Pass
2	3	Pass
3	6	Pass
4	4	Pass
9	10	fail
6	8	fail
9	5	fail
8	7	fail
10	8	fail

$\Rightarrow$

$X_1 = (x_1, x_2) = \{(4,1), (2,4), (2,3), (3,6), (4,4)\}$

$X_2 = (x_1, x_2) = \{(9,10), (6,8), (9,5), (8,7), (10,8)\}$

$X_1 \Rightarrow$ samples for pass class

$X_2 \Rightarrow$ samples for fail class

Step 1: Calculate Scatter Matrix

Step-1 Compute within class scatter matrix (S_W) ② AKS

$$S_W = S_1 + S_2$$

where, S_1 ⇒ is the covariance matrix for X_1 for class pass

S_2 ⇒ is the covariance matrix for X_2 for class fail.

$$S_1 = \frac{1}{n} \sum_{x \in X_c} (x - \mu_1)(x - \mu_1)^T$$

Where μ_1 is the mean of x_1 of class yes which is computed as

$$\mu_1 = \left\{ \frac{4+2+2+3+4}{5}, \frac{1+4+3+6+4}{5} \right\}$$

$$\mu_1 = \{3.00 \quad 3.60\}$$

Similarly $\mu_2 = \{8.4 \quad 7.60\}$

Calculate Mean Correct

Now: $X_1 = \begin{Bmatrix} 4 & 2 & 2 & 3 & 4 \\ 1 & 4 & 3 & 6 & 4 \end{Bmatrix}$ $\mu_1 = \begin{Bmatrix} 3 \\ 3.6 \end{Bmatrix}$

$(X_1 - \mu_1) = \begin{Bmatrix} 1 & -1 & -1 & 0 & 1 \\ -2.6 & 0.4 & -0.6 & 2.4 & 0.4 \end{Bmatrix}$ ⇒ Mean Subtracted Sample for X_1 for class Yes.

Now, for each x, we are going to calculate $(x-\mu_1)(x-\mu_1)^T$. So we will get "5" such matrice in this case.

1) $\begin{bmatrix} 1 \\ -2.6 \end{bmatrix} \begin{bmatrix} 1 & -2.6 \end{bmatrix} = \begin{bmatrix} 1 & -2.6 \\ -2.6 & 6.76 \end{bmatrix}$ — ①

2) $\begin{bmatrix} -1 \\ 0.4 \end{bmatrix} \begin{bmatrix} -1 & 0.4 \end{bmatrix} = \begin{bmatrix} 1 & -0.4 \\ -0.4 & 0.16 \end{bmatrix}$ — ②

④ AKS

$$\begin{bmatrix} -1 \\ 0.6 \end{bmatrix} \begin{bmatrix} -1 & -0.6 \end{bmatrix} = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.36 \end{bmatrix} \quad \text{--- (3)}$$

$$\begin{bmatrix} 0 \\ 2.4 \end{bmatrix} \begin{bmatrix} 0 & 2.4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5.76 \end{bmatrix} \quad \text{--- (4)}$$

$$\begin{bmatrix} 1 \\ 0.4 \end{bmatrix} \begin{bmatrix} 1 & 0.4 \end{bmatrix} = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 0.16 \end{bmatrix} \quad \text{--- (5)}$$

Adding ① ② ③ ④ and ⑤ and taking average we get covariance matrix S_1.

$$S_1 = \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 2.6 \end{bmatrix}$$

Now, Between Class Scatter

Similarly for x_2 for class false, the covariance matrix is given by (5)

$$S_2 = \begin{bmatrix} 1.84 & -0.04 \\ -0.04 & 2.64 \end{bmatrix} \quad \& \quad \mu_2 = [8.4 \quad 7.6]$$

$$\therefore \quad S_W = S_1 + S_2$$

$$= \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 2.6 \end{bmatrix} \begin{bmatrix} 1.84 & -0.04 \\ -0.04 & 2.64 \end{bmatrix}$$

$$= \begin{bmatrix} 2.64 & -0.44 \\ -0.44 & 5.28 \end{bmatrix} \Rightarrow \text{within class scatter Matrix } S_W$$

We have calculated, x_1, x_2, μ_1, μ_2, S_1, S_2 and S_W.

Now we need to calculate $S_B \Rightarrow$ Between class scatter matrix.

Step 2: Calculate SB

(6)

STEP:2: Calculate S_B.

$$S_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T$$

$$= \left[\begin{pmatrix} 3 \\ 3.6 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix}\right] \left[\begin{pmatrix} 3 \\ 3.6 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix}\right]^T$$

$$= \begin{pmatrix} -5.4 \\ -4 \end{pmatrix} \begin{pmatrix} -5.4 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 29.16 & 21.6 \\ 21.6 & 16.00 \end{pmatrix} \Rightarrow S_B$$

Now $S_W = \begin{bmatrix} 2.64 & -0.44 \\ -0.44 & 5.28 \end{bmatrix}$, $S_B = \begin{pmatrix} 29.16 & 21.6 \\ 21.6 & 16.00 \end{pmatrix}$

Step 3: Find Best LDA Projection

STEP-3: find the Best LDA Projection Vector. (7) (8)

We find this using Eigen Vector having largest Eigen value.

We calculate this using formula:

$$S_W^{-1} S_B V = \lambda V \qquad (9)$$

- λ → eigen value (scalar quantity)
- V → eigen vector for corresponding eigen value. (vector quantity)

$$\Rightarrow \left| S_W^{-1} S_B V - \lambda V \right| = 0$$

$$\left| S_W^{-1} S_B - \lambda I \right| = 0$$

I → Identity matrix.

$$\left| \begin{pmatrix} 0.384 & 0.032 \\ 0.032 & 0.192 \end{pmatrix} \begin{pmatrix} 29.16 & 21.6 \\ 21.6 & 16.0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 11.89-\lambda & 8.81 \\ 5.08 & 3.76-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (11.89-\lambda)(3.76-\lambda) - 5.08 \times 8.81 = 0$$

$$\Rightarrow 44.7 - 11.89\lambda - 3.76\lambda + \lambda^2 - 44.7 = 0$$

$$\Rightarrow \lambda^2 - 15.76\lambda = 0$$

$$\Rightarrow \lambda(\lambda - 15.76) = 0$$

$\Rightarrow \lambda = 0$ or $\lambda = 15.76$ → two eigen values we get.

Put highest eigen value in eq (a) on page 7.

Note: we take bigger eigen value to reduce dimention

(9.)

$$\Rightarrow \begin{bmatrix} 11.89 & 8.81 \\ 5.08 & 3.76 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 15.65 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$(S_W^{-1} \cdot S_B)$ — Eigen vector we need to find — higest Eigen value.

we get after solving above eq:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0.91 \\ 0.39 \end{bmatrix} \text{ — Solution}$$

Note: Inverse of Matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$S_W = \begin{bmatrix} 2.64 & -0.44 \\ -0.44 & 5.28 \end{bmatrix}, \quad S_W^{-1} = \frac{1}{13.74} \begin{bmatrix} 5.28 & 0.44 \\ 0.44 & 2.64 \end{bmatrix} = \begin{bmatrix} 0.384 & 0.032 \\ 0.032 & 0.192 \end{bmatrix}$$

STEP:4 Dimention Reduction:

$Y = W^T X$ } → input data samples

W → Projection matrix or higest eigen vector

$\therefore Y = \begin{bmatrix} 0.91 & 0.39 \end{bmatrix}_{1\times2} \begin{bmatrix} 4 & 2 & 2 & 3 & 4 \\ 1 & 4 & 3 & 6 & 4 \end{bmatrix}_{2\times5}$

$Y = \begin{bmatrix} 4.03 & 3.38 & 2.99 & 5.07 & 5.2 \end{bmatrix}$

∴ we see 2-D data sample is changed into 1-D data samples.

Question

Sub 1	Sub2	Status
7	9	Qualified
6	8	Qualified
2	4	Fail
3	1	Fail

Question

Customer ID	Age	Income (USD)	Class
1	30	40	A
2	22	25	B
3	45	80	A
4	52	65	A
5	18	18	B
6	38	50	B
7	60	10	A

Mean

For Class A:

$$\mathbf{m}_A = \left(\frac{30+45+52+60}{4}, \frac{40+80+65+10}{4}\right)$$

$$\mathbf{m}_A = \left(\frac{187}{4}, \frac{195}{4}\right)$$

$$\mathbf{m}_A = (46.75, 48.75)$$

For Class B:

$$\mathbf{m}_B = \left(\frac{22+18+38}{3}, \frac{25+18+50}{3}\right)$$

$$\mathbf{m}_B = \left(\frac{78}{3}, \frac{93}{3}\right)$$

$$\mathbf{m}_B = (26, 31)$$

Sw and Sb

$$\mathbf{S}_w = \mathbf{S}_{w_A} + \mathbf{S}_{w_B}$$

$$\mathbf{S}_{w_A} = \begin{pmatrix} 272.25 & 144.375 \\ 144.375 & 76.5625 \end{pmatrix}$$

$$\mathbf{S}_{w_B} = \begin{pmatrix} 16 & 24 \\ 24 & 36 \end{pmatrix}$$