Tutorial Sheet 9 Solutions

1) a)

$$\begin{cases} x \equiv 5 \pmod{7} \\ x \equiv 4 \pmod{9} \end{cases}$$

$a_1 = 5, \quad a_2 = 4$

$m_1 = 7, \quad m_2 = 9.$

$M = m_1 \cdot m_2 = 7 \cdot 9 \Rightarrow \boxed{M = 63}$

$M_1 = \frac{M}{m_1} \Rightarrow \boxed{M_1 = 9}$

$M_2 = \frac{M}{m_2} \Rightarrow \boxed{M_2 = 7}$

$M_1^{-1} \pmod{m_1} \Rightarrow 9 \times \boxed{} \equiv 1 \pmod{5}$ $\qquad 9 \equiv 4 \pmod{5}$

$\Rightarrow 4 \times \boxed{4} \equiv 1 \pmod{5}$

$\boxed{M_1^{-1} = 4}$

$M_2^{-1} \pmod{m_2} \Rightarrow 7 \times \boxed{4} \equiv 1 \pmod{9}$

$\Rightarrow \boxed{M_2^{-1} = 4}$

$$\begin{aligned} x &\equiv a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} \pmod{M} \\ &\equiv 5 \times 9 \times 4 + 4 \times 7 \times 4 \pmod{63} \\ &\equiv 180 + 112 \pmod{63} \\ &\equiv 54 + 49 \pmod{63} \\ &\equiv 103 \pmod{63} \\ &\equiv \boxed{40} \pmod{63} \end{aligned}$$

b)

$$\begin{cases} x \equiv 1 \pmod{2} \\ x \equiv 2 \pmod{3} \\ x \equiv 3 \pmod{7} \end{cases}$$

$a_1 = 1, \; a_2 = 2, \; a_3 = 3$

$m_1 = 2, \; m_2 = 3, \; m_3 = 7.$

$M = m_1 m_2 m_3 \Rightarrow \boxed{M = 42}$

$\boxed{M_1 = 21}, \boxed{M_2 = 14}, \boxed{M_3 = 6}$

$M_1^{-1} \pmod{m_1} \qquad 21 \times \boxed{} \equiv 1 \pmod{2}$

$\Rightarrow 1 \times \square \equiv 1 \pmod 2$

$\boxed{M_1^{-1} = 1}$

$M_2^{-1} \pmod{m_2}$

$14 \times \square \equiv 1 \pmod 3$ $\qquad 14 \equiv 2 \pmod 3$

$\Rightarrow 2 \times \boxed{?} \equiv 1 \pmod 3$

$\Rightarrow \boxed{M_2^{-1} = 2}$

$M_3^{-1} \pmod{m_3}$ $\qquad 6 \times \boxed{?} \equiv 1 \pmod 7$

$\boxed{M_3^{-1} = 6}$ $\qquad \because 6 \times 6 \equiv 1 \pmod 7$

$\therefore x \equiv a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} + a_3 M_3 M_3^{-1} \pmod M$

$\equiv 1 \times 21 \times 1 + 2 \times 14 \times 2 + 3 \times 6 \times 6 \pmod{42}$

$\equiv 21 + 56 + 108 \pmod{42}$

$\equiv 21 + 14 + 24 \pmod{42}$

$\equiv 59 \pmod{42}$

$\equiv \boxed{17} \pmod{42}.$

2) $\begin{cases} x \equiv 2 \pmod 3 \\ x \equiv 1 \pmod 4 \\ x \equiv 1 \pmod 5 \end{cases}$

$a_1 = 2,\ a_2 = 1,\ a_3 = 1$

$m_1 = 3, m_2 = 4,\ m_3 = 5.$

$\boxed{M = 60}$

$\boxed{M_1 = 20}$, $\boxed{M_2 = 15}$, $\boxed{M_3 = 12}$

$20 \times \square \equiv 1 \pmod 3$

$\Rightarrow 2 \times \square \equiv 1 \pmod 3$

$\boxed{M_1^{-1} = 2}$

$15 \times \square \equiv 1 \pmod 4$

$\Rightarrow 3 \times \square \equiv 1 \pmod 4$

$\Rightarrow \boxed{M_2^{-1} = 3}$

$12 \times \square \equiv 1 \pmod 5$

$\Rightarrow 2 \times \square \equiv 1 \pmod 5$

$\Rightarrow \boxed{M_3^{-1} = 3}$

$x \equiv 2 \times 20 \times 2 + 15 \times 3 + 12 \times 3 \pmod{60}$

$\equiv \underline{20} + 45 + 36 \pmod{60}$

$\equiv \boxed{41} \pmod{60}$

3) $^{10}C_3 = \dfrac{10 \times 9 \times 8}{3 \times 2 \times 1} = \boxed{120}$

4) $^{10}P_3 = \dfrac{10!}{7!} = 10 \times 9 \times 8 = \boxed{720}$

5) a) $10!$ b) $9!$

6) $8! \times 3!$

7) a) 10^3 b) $^{10}P_3$

8) $^{100}C_{50}$

9)
$$\left\{\begin{array}{lcl} F_2 = F_1 + F_0 & \xrightarrow{\times x^2} & F_2 x^2 = F_1 x^2 + F_0 x^2 \\ F_3 = F_2 + F_1 & \xrightarrow{\times x^3} & F_3 x^3 = F_2 x^3 + F_1 x^3 \\ \vdots & + & \vdots \\ F_n = F_{n-1} + F_{n-2} & \xrightarrow{\times x^n} & F_n x^n = F_{n-1} x^n + F_{n-2} x^n \end{array}\right.$$

$$\overline{\sum_{k=2}^{n} F_k x^k = x \sum_{k=1}^{n-1} F_k x^k + x^2 \sum_{k=0}^{n-2} F_k x^k}$$

Let $\sum_{k=0}^{n} F_k x^k = A_n(x)$ and $\lim_{n \to \infty} A_n(x) = A(x)$

$$\Rightarrow \sum_{k=0}^{n} F_k x^k - F_0 - F_1 x = x\left(\sum_{k=0}^{n} F_k x^k - F_0\right) + x^2 \sum_{k=0}^{n-2} F_k x^k$$

$$\Rightarrow A(x) - F_0 - F_1 x = x\left(A(x) - F_0\right) + x^2 \cdot A(x)$$

$$\Rightarrow A(x)\left[1 - x - x^2\right] = F_0 + F_1 x - F_0 x$$

$$\Rightarrow \boxed{A(x) = \frac{1}{1 - x - x^2}}$$

since $F_0 = F_1 = 1$.

10)

// $a_n = 2a_{n-1} + 3a_{n-2} + 2 \quad \forall n \geq 2.$

and $a_0 = 1,$
$a_1 = 2.$

$$a_2 = 2a_1 + 3a_0 + 2$$
$$= 2 \times 2 + 3 + 2$$

$\boxed{a_2 = 9}$

$$a_3 = 2a_2 + 3a_1 + 2^2$$
$$= 2 \times 9 + 3 \times 2 + 4$$

$\boxed{a_3 = 28}$

$$\therefore \quad 3a_3 + 2a_2 - a_1 = 3 \times 28 + 2 \times 9 - 2$$
$$= \boxed{100}$$