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Tutorial Sheet 1 Solutions

1. **Solution:** There are four and eight reduced row echelon forms of a 2×2 and a 3×3 matrices, respectively. These matrices can be obtained by considering the cases based on the ranks and the position of the leading entries.
For a matrix of order 2×2:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For a matrix of order 3×3:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & x & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $x, y \in \mathbb{R}$.

2. **Solution:**

(a) **Step-1:** Use elementary row operations: $R_3 \to R_3 + R_1$ and $R_4 \to R_4 - R_1$, we get

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & 6 & 2 \\ -1 & 2 & 4 & 3 \\ 1 & 2 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & 6 & 2 \\ 0 & 1 & 6 & 6 \\ 0 & 3 & -3 & -1 \end{bmatrix}$$

Step-2: Now use $R_3 \to R_3 - \frac{1}{5}R_2$ and $R_4 \to R_4 - \frac{3}{5}R_2$, we get

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & 6 & 2 \\ 0 & 1 & 6 & 6 \\ 0 & 3 & -3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & 6 & 2 \\ 0 & 0 & \frac{24}{5} & \frac{28}{5} \\ 0 & 0 & \frac{-33}{5} & \frac{-11}{5} \end{bmatrix}$$

Step-3: Now use $R_4 \to R_4 + \frac{33}{24}R_3$, we get

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & 6 & 2 \\ 0 & 0 & \frac{24}{5} & \frac{28}{5} \\ 0 & 0 & \frac{-33}{5} & \frac{-11}{5} \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & 6 & 2 \\ 0 & 0 & \frac{24}{5} & \frac{28}{5} \\ 0 & 0 & 0 & \frac{11}{2} \end{bmatrix}$$

Last matrix is in the row echelon form.

To find reduced row echelon form

Step-4: **Make all leading entries 1.** Use operation $R_2 \to \frac{1}{5}R_2, \quad R_3 \to \frac{5}{24}R_3, \quad R_4 \to \frac{2}{11}R_4$, we get

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & 6 & 2 \\ 0 & 0 & \frac{24}{5} & \frac{28}{5} \\ 0 & 0 & 0 & \frac{-11}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 6/5 & 2/5 \\ 0 & 0 & 1 & 7/6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step-5: **Make 0s above all leading entries.** Use operation $R_1 \to R_1 + R_2$, we get

$$\begin{bmatrix} 1 & 0 & 16/5 & 17/5 \\ 0 & 1 & 6/5 & 2/5 \\ 0 & 0 & 1 & 7/6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now use $R_1 \to R_1 - \frac{16}{5}R_3$ and $R_2 \to R_2 - \frac{6}{5}R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 7/6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now use $R_1 \to R_1 + \frac{1}{3}R_4, \quad R_2 \to R_2 + R_4, \quad R_3 \to R_3 - \frac{7}{6}R_4$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is the required reduced row echelon form. The rank of the matrix is 4.

(b) **Solution:** Proceed like in part (a). Row echelon form is not unique so you may get different forms. But reduced row echelon form is unique which is given as

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of the matrix is 2.

(c) **Solution:** Proceed like in part (a). Row echelon form is not unique so you may get different forms. But reduced row echelon form is unique which is given as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The rank of the matrix is 4.

3. **Solution:** Construct the large augmented matrix that describes both systems and determine the reduced row echelon form as follows.

$$\left[\begin{array}{ccc|c|c} 1 & -1 & 3 & 0 & 3 \\ 2 & -1 & 4 & 1 & 3 \\ -1 & 2 & -4 & 2 & -4 \end{array}\right]$$

Use operations $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$, we get

$$\left[\begin{array}{ccc|c|c} 1 & -1 & 3 & 0 & 3 \\ 0 & 1 & -2 & 1 & -3 \\ 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

. Now use $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - R_2$, we get

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & -3 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

. Now use $R_1 \to R_1 - R_3$ and $R_2 \to R_2 + 2R_3$, we get

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

.

The solutions to the both systems of equations are given by the last two columns of the reduced roe echelon form. They are $x_1 = 0, x_2 = 3, x_3 = 1$ and $x_1 = -2, x_2 = 1, x_3 = 2$.

4. **Solution:**

   (a) Find the reduced row echelon form of augmented matrix $[A|b]$ which is given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

   Solution is $x = y = z = 1$.

   (b) Reduced row echelon form of augmented matrix $[A|b]$ is given by

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

   In the reduced row echelon form there are two free variables hence infinitely many solutions (rank=2, no. of variables=4 so no of free variables= 4-2). Here we get

$$x - y = 0, \quad z + w = 0.$$

Take $y = k_1$ and $w = k_2,;\ k_1,\ k_2 \in \mathbb{R}$. Then solution is given by $\left\{ \begin{bmatrix} k_1 \\ k_1 \\ -k_2 \\ k_2 \end{bmatrix}, k_1, k_2 \in \mathbb{R} \right\}$.

(c) Reduced row echelon form of augmented matrix $[A|b]$ is given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Here $rank(A) = 2 \neq 3 = rank(A|b)$. Given system is inconsistent.

(d) Reduced row echelon form of augmented matrix $[A|b]$ is given by

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1/2 \\ 0 & 1 & 0 & 2 & 1/2 \\ 0 & 0 & 1 & 4 & 5/2 \end{array}\right]$$

Here $rank(A) = rank(A|b) = 3 < 4$, hence infinitely many solutions. We get equivalent system of equations:

$$w + z = 1/2,\ x + 2z = 1/2,\ y + 4z = 5/2.$$

Take $z = k_1$, where k_1 be any arbitrary real number. Then solution is given by $\left\{ \begin{bmatrix} \frac{1}{2} - k_1 \\ \frac{1}{2} - 2k_1 \\ \frac{5}{2} - 4k_1 \\ k_1 \end{bmatrix}, k_1, k_2 \in \mathbb{R} \right\}$.

5. **Solution:**

(a) **Step-1:** Find augmented matrix:

$$\left[\ A \mid b\ \right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & \lambda & 4 \\ 2 & 3 & 2\lambda & k \end{array}\right]$$

Step-2: Find row echelon form of augmented matrix which is given as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & \lambda - 1 & 1 \\ 0 & 0 & \lambda - 1 & k - 7 \end{array}\right]$$

Step-3: Consider all three possibilities of solutions for the equivalent system

<u>Case-I:</u> If $\lambda = 1,\ k \neq 7$, then no solution.

<u>Case-II:</u> If $\lambda \neq 1$ then unique solution.

$$x = k - 5 - \frac{k-7}{\lambda - 1},\ y = 8 - k,\ z = \frac{k-7}{\lambda - 1}.$$

Case-III: If $\lambda = 1$, $k = 7$, then infinitely many solutions: $\left\{ \begin{bmatrix} 2-k_1 \\ 1 \\ k_1 \end{bmatrix} : k_1 \in \mathbb{R} \right\}$.

(b) **Step-1:** Find augmented matrix:

$$\left[\begin{array}{c|c} A & b \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 1 & 2 & \lambda & 5 \\ 1 & 2 & 4 & k \end{array}\right]$$

Step-2: Find row echelon form of augmented matrix which is given as

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & \lambda-2 & 2 \\ 0 & 0 & 4-\lambda & k-5 \end{array}\right]$$

Step-3: Consider all three possibilities of solutions for the equivalent system
Case-I: If $\lambda = 4$, $k \neq 5$, then no solution.
Case-II: If $\lambda \neq 4$ then unique solution.

$$x = 6-k,\ y = 2 + \frac{k-5}{\lambda-4}(\lambda-2),\ z = \frac{k-5}{4-\lambda}.$$

Case-III: If $\lambda = 4$, $k = 5$, then infinitely many solutions: $\left\{ \begin{bmatrix} 1 \\ 2-2k_1 \\ k_1 \end{bmatrix} : k_1 \in \mathbb{R} \right\}$.

6. **Solution:** Let y be a non-trivial solution of $A^2x = 0$. So we get, $A^2y = 0$. Now if $Ay = 0$ then y is a non-trivial solution of $Ax = 0$. If $Ay \neq 0$ then Ay is a non-trivial solution of $Ax = 0$.