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	CO1	CO2	CO3	CO4	CO5
Q1					✓
Q2					✓
Q3					✓
Q4					✓
Q5					✓

Tutorial Sheet 10 Solution

1) Consider the linear differential equation

$$x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = 0.$$

(a) Show that x^3 and $|x^3|$ are two linearly independent solutions of the differential equation on $x \in (-\infty, \infty)$.

Solution: Suppose $c_1, c_2 \in \mathbb{R}$ and

$$c_1 y_1(x) + c_2 y_2(x) = 0, \quad \forall x \in (-\infty, \infty),$$

then

$$c_1 x^3 + c_2 |x^3| = 0, \quad \forall x \in (-\infty, \infty).$$

Now take $x = 1$, we get

$$c_1 + c_2 = 0.$$

For $x = -1$, we get

$$-c_1 + c_2 = 0.$$

Solving these two we get, $c_1 = c_2 = 0$. Hence x^3 and $|x^3|$ are two linearly independent.

(b) x^3 and $|x^3|$ are two linearly independent solutions of the differential equation but $W(x^3, |x^3|) = 0, \forall x \in \mathbb{R}$. Does it violate any result? Explain.

Solution: We know that

$$W(y_1, y_2)(x) = \det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}.$$

Thus,

$$W(x^3, |x^3|) = \begin{cases} x^3(3x^2) - 3x^2(x^3) & x \geq 0, \\ -x^3(3x^2) - x^3(-3x^2) & x < 0. \end{cases}$$

$$\Rightarrow W(x^3, |x^3|) = 0, \quad \forall x \in \mathbb{R}.$$

Note that $a_0(x) = x^2$ is zero at $x = 0$. The theorem stating that wronskian of linearly independent solutions is non-zero, is not applicable here.

(c) x^2 and x^3 are also two linearly independent solutions of the differential equation. Can we write general solution of the differential equation in terms of these solutions?

Solution: No. None of the linear combinations $c_1x^3 + c_2|x^3|$ or $C_1x^2 + C_2x^3$ is a general solution of the given DE in the interval $(-\infty, \infty)$ because if $C_1x^2 + C_2x^3$ is a general solution on the interval $(-\infty, \infty)$ we should be able to find constants C_1 and C_2 such that

$$x^3 + |x^3| = C_1x^2 + C_2x^3 \quad \forall x \in (-\infty, \infty), \quad (1)$$

which is not possible. Because if we take $x = 1$ in equation (1), we get $C_1 + C_2 = 2$. Taking $x = -1$ in equation (1), we get $C_1 - C_2 = 0$. Solving these two for C_1 and C_2 , we get $C_1 = C_2 = 1$, i.e.,

$$x^3 + |x^3| = x^2 + x^3 \quad \forall x \in (-\infty, \infty) \Rightarrow |x^3| = x^2 \quad \forall x \in (-\infty, \infty),$$

which is not possible. Similarly we can show that $c_1x^3 + c_2|x^3|$ cannot be a general solution.

2) Use reduction of order method to find the second linearly independent solution of the following differential equations

(a) $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$; $y_1(x) = x$.

Solution: Let $y_2(x) = y_1(x)v(x)$ is the second linearly independent solution for a suitable function $v(x)$. Then $y_1(x) = x$ implies

$$y_2 = xv \Rightarrow \frac{dy_2}{dx} = x\frac{dv}{dx} + v \Rightarrow \frac{d^2y_2}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}.$$

Substituting the expressions for y_2 , $\frac{dy_2}{dx}$ and $\frac{d^2y_2}{dx^2}$ into the given ODE, we get

$$(x-1)\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - x\left(x\frac{dv}{dx} + v\right) + vx = 0$$

$$\Rightarrow x(x-1)\frac{d^2v}{dx^2} + \{2(x-1) - x^2\}\frac{dv}{dx} = 0$$

$$\text{Taking } \frac{dv}{dx} = w \Rightarrow \frac{d^2v}{dx^2} = \frac{dw}{dx}.$$

Using this we get a first order ODE

$$x(x-1)\frac{dw}{dx} + \{2(x-1) - x^2\}w = 0$$

$$\Rightarrow \frac{dw}{w} + \frac{\{2(x-1) - x^2\}}{x(x-1)}dx = 0$$

$$\Rightarrow \frac{dw}{w} + \frac{2}{x}dx - \frac{1}{x-1}dx - dx = 0$$

On integrating,

$$\ln|w| + 2\ln|x| - \ln|x-1| - x = \ln|c| \Rightarrow \text{for } c = 1, w = \frac{e^x(x-1)}{x^2}.$$

$$\int dv = \int \left(\frac{e^x}{x} - \frac{e^x}{x^2} \right) dx \Rightarrow v = \frac{e^x}{x} + c_1,$$

where c_1 is an arbitrary constant. Thus for $c_1 = 0$,

$$y_2 = vx = e^x,$$

is a second linearly independent solution. The general solution is

$$y(x) = Ax + Be^x,$$

where A and B are two arbitrary constants.

(b) $x^2 \frac{d^2y}{dx^2} - (2a - 1)x \frac{dy}{dx} + a^2y = 0$; $a \neq 0$, $x > 0$, $y_1(x) = x^a$.

Solution: Proceed like in part (a). Second solution is $y_2 = x^a \log x$.

(c) $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + 2y = 0$; $y_1(x) = x \sin(\log x)$.

Solution: Proceed like in part (a). Second solution is $y_2 = -x \cos(\log x)$.

3) Find the second order differential equation corresponding to given linearly independent solutions

(a) $y_1 = \cos 2\pi x$, $y_2 = \sin 2\pi x$.

Solution: The general solution is

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 \cos 2\pi x + c_2 \sin 2\pi x$$

Clearly the roots of auxiliary equation (A. E.) of the given differential equation are imaginary, i.e., when $m = a \pm ib$, solution is

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx).$$

On comparing $a = 0$ and $b = 2\pi$, i.e., roots are $\pm 2\pi i$. Therefore the A. E. is

$$(m - 2\pi i)(m + 2\pi i) = 0 \Rightarrow m^2 + 4\pi^2 = 0$$

Thus, the differential equation is

$$\frac{d^2y}{dx^2} + 4\pi^2 y = 0.$$

(b) $y_1 = e^{-\sqrt{2}x}$, $y_2 = xe^{-\sqrt{2}x}$.

Solution: The general solution is

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{-\sqrt{2}x} + c_2 x e^{-\sqrt{2}x} = (c_1 + c_2 x) e^{-\sqrt{2}x}$$

Clearly the roots of A. E. of differential equation are repeated and here $m = -\sqrt{2}$. Therefore the A. E. is

$$(m + \sqrt{2})(m + \sqrt{2}) = 0 \Rightarrow m^2 + 2\sqrt{2}m + 2 = 0.$$

Thus, the differential equation is

$$\frac{d^2y}{dx^2} + 2\sqrt{2} \frac{dy}{dx} + 2y = 0.$$

(c) $y_1 = e^{(-1+i\sqrt{2})x}$, $y_2 = e^{(-1-i\sqrt{2})x}$.

Solution:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0.$$

4) Solve the IVP's

(a) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$; $y(-1) = e$, $y'(-1) = -\frac{e}{4}$.

Solution: Auxiliary equation is

$$m^2 - 2m - 3 = 0 \Rightarrow (m - 3)(m + 1) = 0 \Rightarrow m = -1, 3.$$

Thus, the general solution is

$$y(x) = c_1e^{-x} + c_2e^{3x}.$$

Now use the initial conditions:

$$y(-1) = e \Rightarrow c_1e + c_2e^{-3} = e,$$

and

$$y'(-1) = -\frac{e}{4} \Rightarrow -c_1e + 3c_2e^{-3} = \frac{-e}{4}.$$

Solving for c_1 and c_2 , we get $c_1 = \frac{13}{16}$ and $c_2 = \frac{3e^4}{16}$. Hence the solution is

$$y(x) = \frac{13}{16}e^{-x} + \frac{3e^4}{16}e^{3x}.$$

(b) $\frac{d^2y}{dx^2} - k^2y = 0$; $k \neq 0$, $y(0) = 1$, $y'(0) = 1$.

Solution: Auxiliary equation is

$$m^2 - k^2 = 0 \Rightarrow (m - k)(m + k) = 0 \Rightarrow m = k, -k.$$

Thus, the general solution is

$$y(x) = c_1e^{kx} + c_2e^{-kx}.$$

Now use the initial conditions:

$$y(0) = 1 \Rightarrow c_1 + c_2 = 1,$$

and

$$y'(0) = 1 \Rightarrow kc_1 - kc_2 = 1.$$

Solving for c_1 and c_2 , we get $c_1 = \frac{k+1}{2k}$ and $c_2 = \frac{k-1}{2k}$. Hence the solution is

$$y(x) = \frac{k+1}{2k}e^{kx} + \frac{k-1}{2k}e^{-kx}.$$

(c) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$; $y(0) = 3$, $y'(0) = -1$.

Solution: Do yourself.

(d) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$; $y(0) = 2$, $y'(0) = -3$.

Solution: Do yourself.

(e) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$; $y(0) = 1$, $y'(0) = -8$, $y''(0) = -4$.

Solution: Do yourself.

5) Solve the following non-homogeneous differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = \frac{e^{-3x}}{x^3}$.

Solution: Here we need to use variation of parameters as method of undetermined coefficients is not applicable. The complementary function is

$$y_c = c_1e^{-3x} + c_2xe^{-3x}.$$

Take $y_1 = e^{-3x}$ and $y_2 = xe^{-3x}$, then

$$W(y_1, y_2) = e^{-6x}.$$

This implies that

$$v_1 = - \int \frac{f(x)y_2}{a_0(x)W(y_1, y_2)} dx = - \int \frac{e^{-3x}}{x^3} \times \frac{xe^{-3x}}{e^{-6x}} dx = \frac{1}{x},$$

similarly

$$v_2 = \int \frac{f(x)y_1}{a_0(x)W(y_1, y_2)} dx = \int \frac{e^{-3x}}{x^3} \times \frac{e^{-3x}}{e^{-6x}} dx = -\frac{1}{2x^2}.$$

Thus, the particular solution is

$$y_p = v_1y_1 + v_2y_2 = \frac{1}{x} \times e^{-3x} - \frac{1}{2x^2}xe^{-3x} = \frac{e^{-3x}}{2x}.$$

Hence the general solution is

$$y = y_c + y_p = c_1e^{-3x} + c_2xe^{-3x} + \frac{e^{-3x}}{2x}.$$