

**Department of Mathematics**  
**Bennett University**  
**EMAT102L**  
**Tutorial Sheet-3 Solutions**

- 1) Prove that the set  $U = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}; a, b \in \mathbb{R} \right\}$ , is a subspace of the vector space  $M_{2 \times 2}(\mathbb{R})$  of  $2 \times 2$  matrices.

**Solution:** We have to show that  $U$  contains zero matrix,  $U$  is closed under addition and under scalar multiplication. Consider the following two arbitrary elements of  $U$ :  $\mathbf{u} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ . We get

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} a+x & 0 \\ 0 & b+y \end{bmatrix}.$$

Observe that  $\mathbf{u} + \mathbf{v}$  is a  $2 \times 2$  diagonal matrix, and it thus an element of  $U$ .  $U$  is closed under addition. Let  $c \in \mathbb{R}$ , we get

$$c\mathbf{u} = c \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ca & 0 \\ 0 & cb \end{bmatrix}.$$

Observe that  $c\mathbf{u}$  is a  $2 \times 2$  diagonal matrix, and it thus an element of  $U$ .  $U$  is closed under scalar multiplication. Thus,  $U$  is a subspace of  $M_{2 \times 2}$ . It is a vector space of matrices.

- 2) Which of the following sets of vectors  $x = (x_1, x_2, x_3)^t$  in  $\mathbb{R}^3$  are subspaces of  $\mathbb{R}^3$ ?

- (a) All  $x$  such that  $x_1 \geq 0$ ,
- (b) All  $x$  such that  $x_1 + 3x_2 = x_3$ ,
- (c) All  $x$  such that  $x_2 = x_1^2$ ,
- (d) All  $x$  such that  $x_1x_2 = 0$ ,
- (e) All  $x$  such that  $x_2$  is rational,
- (f) All  $x$  such that  $x_1^2 + x_2^2 + x_3^2 \leq 1$ .

**Solution:**

- (a) False. It is not closed with respect to scalar multiplication. For example, for any  $x = (x_1, x_2, x_3)^t$  with  $x_1 > 0$ ,  $-1 \cdot x = (-x_1, -x_2, -x_3)^t$  does not belong to the set. Take  $x = (1, 1, 1)^t$  from the set, for scalar  $\alpha = -1$ ,  $\alpha x = -1(1, 1, 1)^t = (-1, -1, -1)^t$  does not belong to the set.
- (b) True. Use the definition.
- (c) False. It is not closed with respect to vector addition. For example,  $x = (1, 1, 1)$  belongs to the set but  $x + x$  does not.
- (d) False. It is not closed with respect to vector addition. For example,  $x = (1, 0, 1)$  and  $y = (0, 1, 1)$  belong to the set but  $x + y$  does not.
- (e) False. It is not closed with respect to scalar multiplication. For example,  $x = (1, 2, 3)^t$  belongs to the set but  $\frac{\pi}{2} \cdot x = (\frac{\pi}{2}, \pi, \frac{3\pi}{2})^t$  does not, because  $\pi$  is a irrational number.

(f) False. It is not closed with respect to scalar multiplication. For example,  $x = (\frac{1}{2}, 0, 0)^t$  belongs to the set but  $5x$  does not.

- 3) Express  $v = [2 \ -5 \ 3]^t$  in a linear combination of  $u_1 = [1 \ -3 \ 2]^t$ ,  $u_2 = [2 \ -4 \ 1]^t$ , and  $u_3 = [1 \ -5 \ 7]^t$ .

**Solution:** We need to find  $c_1, c_2, c_3 \in \mathbb{R}$  such that

$$\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}.$$

This leads to system of equations

$$c_1 + 2c_2 + c_3 = 2, -3c_1 - 4c_2 - 5c_3 = -5, 2c_1 + c_2 + 7c_3 = 3.$$

On solving these equations we get,  $c_1 = \frac{1}{4}$ ,  $c_2 = \frac{3}{4}$ ,  $c_3 = \frac{1}{4}$ . Thus

$$\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}.$$

- 4) Express  $M = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}$  as a linear combination of  $A, B, C$  where  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , and  $C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$ .

**Solution:** Proceed like in above problem to get  $M = 2A + 3B - C$ .

- 5) Show that the vectors  $v_1 = [1 \ 1 \ 1]^t$ ,  $v_2 = [1 \ 2 \ 3]^t$ ,  $v_3 = [1 \ 5 \ 8]^t$  span  $\mathbb{R}^3$ .

**Solution:** Let  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  be any element of  $\mathbb{R}^3$ . Then  $x \in \text{span}(v_1, v_2, v_3)$  if there exist  $\alpha, \beta, \gamma \in \mathbb{R}$  such that  $x = \alpha v_1 + \beta v_2 + \gamma v_3$ , that is,

$$\alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 8 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The above system is uniquely solvable (check this?). Since the above system is solvable for any  $x_1, x_2, x_3$ , we have that any element of  $\mathbb{R}^3$  can be written as a linear combination of  $v_1, v_2$  and  $v_3$ . Thus  $\mathbb{R}^3 \subseteq \text{span}(v_1, v_2, v_3)$ . But we know that  $\text{span}(v_1, v_2, v_3)$  is a subspace of  $\mathbb{R}^3$ , that means  $\text{span}(v_1, v_2, v_3) \subseteq \mathbb{R}^3$ . Thus  $\text{span}(v_1, v_2, v_3) = \mathbb{R}^3$ .

6) Determine whether or not the matrices  $v_1$  and  $v_2$  are linearly dependent where  $v_1 = \begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \end{bmatrix}$

and  $v_2 = \begin{bmatrix} -4 & -12 & 16 \\ -20 & 0 & 4 \end{bmatrix}$ .

**Solution:** Find  $c_1, c_2 \in \mathbb{R}$  such that

$$c_1 \begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} -4 & -12 & 16 \\ -20 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 - 4c_2 & 3c_1 - 12c_2 & -4c_1 + 16c_2 \\ 5c_1 - 20c_2 & 0 & -c_1 + 4c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow c_1 - 4c_2 = 0, 3c_1 - 12c_2 = 0, -4c_1 + 16c_2 = 0, 5c_1 - 20c_2 = 0, -c_1 + 4c_2 = 0.$$

This system of equations has infinitely many solutions  $c_1 = 4k, c_2 = k$ , where  $k \in \mathbb{R}$ . Take  $k = 1$ , then  $C_1 = 4$ , and  $c_2 = 1$ , we see

$$4 \begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \end{bmatrix} + 1 \begin{bmatrix} -4 & -12 & 16 \\ -20 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus  $v_1$  and  $v_2$  are linearly dependent.

7) Determine whether the set  $\{[1, 2, 0]^t, [0, 1, -1]^t, [1, 1, 2]^t\}$  is linearly independent in  $\mathbb{R}^3$ .

**Solution:** For  $c_1, c_2, c_3 \in \mathbb{R}$ , we examine the identity

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The identity leads to the following system of linear equations:

$$c_1 + c_3 = 0, 2c_1 + c_2 + c_3 = 0, -c_2 + 2c_3 = 0.$$

This system has unique solution  $c_1 = c_2 = c_3 = 0$ . Thus the set is linearly independent.

8) Show that the vectors  $v_1 = [1 + i \ 2i]^t$  and  $v_2 = [1 \ 1 + i]^t$ , in  $\mathbb{C}^2$  are linearly dependent over the field  $\mathbb{C}$  but linearly independent over the field  $\mathbb{R}$ .

**Solution:**  $v_1$  and  $v_2$  are linearly dependent over  $\mathbb{C}$  because  $v_1 - (1 + i)v_2 = 0$ . But  $v_1$  and  $v_2$  are linearly independent over  $\mathbb{R}$ , because for any  $\alpha, \beta \in \mathbb{R}$  such that

$$\alpha v_1 + \beta v_2 = 0 \Rightarrow \begin{bmatrix} 1 + i & 1 \\ 2i & 1 + i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This implies  $\alpha = 0$  and  $\beta = 0$ . Thus  $\alpha, \beta \in \mathbb{R}$  such that  $\alpha v_1 + \beta v_2 = 0$  implies that  $\alpha = 0$  and  $\beta = 0$ . Therefore by definition  $v_1$  and  $v_2$  are linearly independent.

9) Determine whether or not each of the following forms a basis of  $\mathbb{R}^3$ :

(a)  $v_1 = [1 \ 2 \ 3]^t, v_2 = [1 \ 3 \ 5]^t, v_3 = [1 \ 0 \ 1]^t, v_4 = [2 \ 3 \ 0]^t$ .

**Solution:** No. The dimension of  $\mathbb{R}^3$  is 3. Any set of vectors more than 3 vectors in  $\mathbb{R}^3$  is linearly dependent. These four vectors are linearly dependent (check). These vectors do not form a basis of  $\mathbb{R}^3$ .

(b)  $v_1 = [1 \ 1 \ 1]^t, v_2 = [1 \ 2 \ 3]^t, v_3 = [2 \ -1 \ 1]^t$ .

**Solution:** Check that these vectors are linearly independent in  $\mathbb{R}^3$ . The dimension of  $\mathbb{R}^3$  is 3. We know that any three linearly independent vectors in  $\mathbb{R}^3$  will form a basis of  $\mathbb{R}^3$ . So, these vectors form a basis of  $\mathbb{R}^3$ .

(c)  $v_1 = [1 \ 1 \ 2]^t, v_2 = [1 \ 2 \ 5]^t, v_3 = [5 \ 3 \ 4]^t$ .

**Solution:** No, as these three vectors are linearly dependent. These vectors do not form a basis of  $\mathbb{R}^3$ .

10) (a) Show that the set  $\{x^2 + 1, 3x - 1, -4x + 1\}$  is linearly independent in  $\mathbb{P}_2(\mathbb{R})$ .

**Solution:** For  $c_1, c_2, c_3 \in \mathbb{R}$ , consider the identity

$$c_1(x^2 + 1) + c_2(3x - 1) + c_3(-4x + 1) = 0.$$

This gives  $c_1x^2 + (3c_2 - 4c_3)x + (c_1 - c_2 + c_3) = 0$ . Such a polynomial can only be zero if each coefficient is zero. Thus we get

$$c_1 = 0, 3c_2 - 4c_3 = 0, c_1 - c_2 + c_3 = 0.$$

This system of equations has the unique solution  $c_1 = c_2 = c_3 = 0$ . Thus polynomials are linearly independent.

(b) Show that the set  $\{x + 1, x - 1, -x + 5\}$  is linearly dependent in  $\mathbb{P}_1(\mathbb{R})$ .

**Solution:** For  $c_1, c_2, c_3 \in \mathbb{R}$ , consider the identity

$$c_1(x + 1) + c_2(x - 1) + c_3(-x + 5) = 0.$$

This gives  $(c_1 + c_2 - c_3)x + (c_1 - c_2 + 5c_3) = 0$ . Such a polynomial can only be zero if each coefficient is zero. Thus we get

$$c_1 + c_2 - c_3 = 0, c_1 - c_2 + 5c_3 = 0.$$

This system has infinitely many solutions,  $c_1 = -2k, c_2 = 3k, c_3 = k$ , where  $k \in \mathbb{R}$ . Take  $k = 1$ , then

$$-2(x + 1) + 3(x - 1) + 1(-x + 5) = 0.$$

Thus polynomials are linearly dependent.

11) What is the span of  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ ?

**Solution:**

$$\begin{aligned} \text{Span} &= \{A_{2 \times 2} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; a, b, c \in \mathbb{R}\} \\ &= \{A = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}; a, b, c \in \mathbb{R}\} \\ &= \{A = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}; A = A^t \& a, b, c \in \mathbb{R}\} \end{aligned}$$

Span is the set of all  $2 \times 2$  symmetric matrices.

12) Prove that the set  $V = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & f \\ 0 & 0 & g \end{bmatrix} \in \mathcal{M}_{3 \times 3}(\mathbb{R}) : a + b + c = 0, a + d + g = 0 \right\}$  is a vector space of  $\mathcal{M}_{3 \times 3}(\mathbb{R})$  and find a basis for it and its dimension.

**Solution:** Use definition (like in problem 1) to show that  $V$  is a subspace of  $\mathcal{M}_{3 \times 3}(\mathbb{R})$ .

For Basis: Let  $\begin{bmatrix} a & b & c \\ 0 & d & f \\ 0 & 0 & g \end{bmatrix} \in V$ . Then  $a + b + c = 0 \Rightarrow a = -b - c$ , and  $a + d + g = 0 \Rightarrow d = -a - g = b + c - g$ . Thus we have

$$\begin{aligned} \begin{bmatrix} a & b & c \\ 0 & d & f \\ 0 & 0 & g \end{bmatrix} &= \begin{bmatrix} -b - c & b & c \\ 0 & b + c - g & f \\ 0 & 0 & g \end{bmatrix} \\ &= b \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + g \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Thus,  $\left\{ \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$  spans  $V$  and these four matrices are linearly independent (check it).

Thus,  $\left\{ \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$  forms a basis for  $V$ .

Dimension of  $V$  is 4.