Tutorial Sheet 5 Solutions

1) a) Yes.

Checking criteria $T: V \longrightarrow W$.

$T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$

$\forall \; \alpha, \beta \in F$

and $v_1, v_2 \in V$

$T(x,y) = (x+y, x-y)$.

Choose $v_1, v_2 \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$

$v_1 = (x_1, y_1)$

$v_2 = (x_2, y_2)$.

LHS: $T(\alpha v_1 + \beta v_2) = T\left(\alpha(x_1,y_1) + \beta(x_2,y_2)\right)$

$= T\left((\alpha x_1, \alpha y_1) + (\beta x_2, \beta y_2)\right)$

$= T\left(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2\right)$

$= \left(\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2 - (\alpha y_1 + \beta y_2)\right)$

$\left(\because T(x,y) = (x+y, x-y)\right)$

RHS $= \alpha T(v_1) + \beta T(v_2)$

$= \alpha T(x_1, y_1) + \beta T(x_2, y_2)$

$= \alpha(x_1+y_1, x_1-y_1) + \beta(x_2+y_2, x_2-y_2)$

$= (\alpha x_1 + \alpha y_1, \alpha x_1 - \alpha y_1) + (\beta x_2 + \beta y_2, \beta x_2 - \beta y_2)$

$= (\alpha x_1 + \alpha y_1 + \beta x_2 + \beta y_2, \alpha x_1 - \alpha y_1 + \beta x_2 - \beta y_2)$

$= LHS.$

$\therefore$ T is a LT.

b) No.

Search counter example.

Observe $T(0,0) = (0,1)$

But $(0,1)$ is <u>not</u> the zero element of $\mathbb{R}^2$.

> We know if T is a LT, then the zero element of V must map to the zero element of W.

c) Yes.

$v_1 = (x_1, y_1, z_1) \in \mathbb{R}^3$, $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$

$\alpha, \beta \in \mathbb{R}$.

LHS $T(\alpha v_1 + \beta v_2)$

$$= T\big(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)\big)$$

$$= T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

$$= \big(\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2 - (\alpha y_1 + \beta y_2) + (\alpha z_1 + \beta z_2)\big)$$

RHS $= \alpha T(v_1) + \beta T(v_2)$

$$= \alpha T(x_1, y_1, z_1) + \beta T(x_2, y_2, z_2)$$

$$= \alpha(x_1+y_1,\ x_1-y_1+z_1) + \beta(x_2+y_2,\ x_2-y_2+z_2)$$

$$= \left(\alpha x_1+\alpha y_1+\beta x_2+\beta y_2,\ \alpha x_1-\alpha y_1+\alpha z_1+\beta x_2-\beta y_2+\beta z_2\right)$$

$$= \text{LHS}.$$

d) No. $\quad T(x,y,z) = (xy,\ yz,\ zx)$

$$T(1,2,0) = (2,0,0)$$

$$3\cdot T(1,2,0) = 3(2,0,0) = (6,0,0)$$

But $T(3\cdot(1,2,0)) = T(3,6,0) = (18,0,0)$

$\therefore$ Since $T(3\cdot(1,2,0)) \neq 3\cdot T(1,2,0)$

$\Rightarrow$ T is not a LT.

e) Yes

f) Yes

Consider $v_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $v_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$\alpha, \beta \in \mathbb{R}$

$$\text{LHS} = T(\alpha v_1 + \beta v_2) = T\left(\alpha\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \beta\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} \alpha a_1+\beta a_2 & \alpha b_1+\beta b_2 \\ \alpha c_1+\beta c_2 & \alpha d_1+\beta d_2 \end{bmatrix}\right)$$

$$= \left(\alpha a_1+\beta a_2+\alpha b_1+\beta b_2,\ \alpha a_1+\beta a_2-\alpha b_1-\beta b_2,\ \alpha a_1+\beta a_2\right)$$

$\therefore$ Since $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a+b,\ a-b,\ a)$

RHS: $\alpha T(v_1) + \beta T(v_2)$

$$= \alpha T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + \beta T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= \alpha(a_1+b_1, a_1-b_1, a_1) + \beta(a_2+b_2, a_2-b_2, a_2)$$

$$= (\alpha a_1+\alpha b_1+\beta a_2+\beta b_2, \alpha a_1-\alpha b_1+\beta a_2-\beta b_2, \alpha a_1+\beta a_2)$$

$= LHS.$

2) $T: \mathbb{R}^3 \xrightarrow{L.T.} \mathbb{R}^3$

$T(1,1,0) = (1,2,3)$

$T(0,1,1) = (1,0,1)$.

$(2,3,1) = \alpha(1,1,0) + \beta(0,1,1) = (\alpha, \alpha+\beta, \beta)$

$\alpha = 2; \beta = 1$.

$\therefore (2,3,1) = 2(1,1,0) + (0,1,1)$

Now $T(2,3,1) = 2T(1,1,0) + T(0,1,1)$

$= 2\cdot(1,2,3) + (1,0,1)$

$= (2,4,6) + (1,0,1) = \underline{\underline{(3,4,7)}}$ Ans.

Similarly,

$(2,0,0) = \alpha(1,1,0) + \beta(0,1,1) = (\alpha, \alpha+\beta, \beta)$

$\Rightarrow \alpha = 2, \alpha+\beta = 0, \beta = 0.$

inconsistent.

$\Rightarrow$ no solution.

$\Rightarrow (2,0,0) \notin \text{Span}\{(1,1,0),(0,1,1)\}$.

$\therefore$ $T(2,0,0)$ cannot be determined with the given data.

3) $(2,3) = \alpha(1,1) + \beta(0,1) = (\alpha, \alpha+\beta)$

$\Rightarrow \alpha = 2$, and $\alpha+\beta = 3$. $\Rightarrow \boxed{\alpha=2, \beta=1}$

$\therefore (2,3) = 2(1,1) + (0,1)$

$\Rightarrow T(2,3) = 2\,T(1,1) + T(0,1)$

$= 2(1,2) + (2,1) = (2,4)+(2,1) = \underline{\underline{(4,5)}}$ Ans.

Yes. Because $\{(1,1),(0,1)\}$ is a basis of $\mathbb{R}^2$.

$(x,y) = \alpha(1,1) + \beta(0,1) = (\alpha, \alpha+\beta)$

$\Rightarrow \boxed{\alpha = x}$, $\alpha+\beta = y$ $\Rightarrow \boxed{\beta = y-x}$

$\therefore T(x,y) = \alpha\,T(1,1) + \beta\,T(0,1)$

$= x(1,2) + (y-x)(2,1)$

$= (x,2x) + (2y-2x, y-x)$

$\boxed{T(x,y) = (2y-x,\ y+x)}$.

4) a) $N(T) = \{(x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0)\}$

$= \{(x,y) \in \mathbb{R}^2 \mid (x+y, x-y) = (0,0)\}$.

$$= \{(0,0)\}.$$

$$\begin{cases} x+y=0 \\ x-y=0 \end{cases} \Rightarrow \boxed{x=y=0}$$

$$\dim(\mathcal{N}(T)) = \text{nullity}(T) = 0.$$

$$\mathcal{R}(T) = \{T(x,y) \mid (x,y)\in\mathbb{R}^2\}.$$

To find range space of T,

1. Plug the standard basis elements of input space
2. Consider the output set.
3. Find out a LI subset of the output set.
4. Then range space is the span of that output set.

NOTE:
Null space is a subspace of the domain
on the other-hand
Range space is a subspace of the codomain.

To find range space of T, compute

$$T(1,0) = (1,1)$$

$$T(0,1) = (1,-1)$$

$$\therefore \mathcal{R}(T) = \text{Span}\{\underbrace{(1,1),\ (1,-1)}_{\text{LI}}\}$$

So $\dim(\mathcal{R}(T)) = \text{rank}(T) = 2.$

b) ✗

c) Hint:

Computation of $Ker(T) = \mathcal{N}(T)$ <u>null space of T</u>:

$$T(x,y,z) = (0,0)$$

$$\Rightarrow (x+y,\ x-y+z) = (0,0)$$

$$\Rightarrow \begin{cases} x+y = 0 \\ x-y+z=0 \end{cases} \Rightarrow x=-y \quad 2x+z=0$$

combining

general sol^n.

choose

$$\boxed{\begin{aligned} x &= t \in \mathbb{R} \\ y &= -t \\ z &= -2t \end{aligned}}$$

$\therefore$ an element of the null space of T will look like $(t,-t,-2t)$.

$$\therefore \mathcal{N}(T) = span\{(1,-1,-2)\}$$

$$nullity(T) = 1.$$

Computation of $\mathcal{R}(T)$ <u>range space</u> of T :

$$T(1,0,0) = (1,1)$$

$$T(0,1,0) = (1,-1)$$

$$T(0,0,1) = (0,1)$$

$$\therefore \mathcal{R}(T) = span\{\underbrace{(1,1),\ (1,-1),\ (0,1)}_{LD}\}$$

$= \text{span}\{(1,1),(1,-1)\}$

$\text{rank}(T) = 2$ Since a basis of $R(T)$ is $\{(1,1),(1,-1)\}$

d) $\times$

e) Null space :

$$T(x,y,z) = (x+y,\ y+z,\ x+2y+z) = (0,0,0)$$

$$\Rightarrow \begin{cases} x+y=0 \\ y+z=0 \\ x+2y+z=0 \end{cases} \Rightarrow \boxed{x=-y=z}$$

general form

$$N(T) = \{(t,-t,t),\ t\in\mathbb{R}\} \quad \boxed{\begin{matrix} x=t & \forall\, t\in\mathbb{R} \\ y=-t & \\ z=t & \end{matrix}}$$

a Basis of $N(T)$

$= \{(1,-1,1)\}$.

$N(T) = \text{span}\{(1,-1,1)\}$.

$\text{nullity}(T) = 1$.

Range space :

$T(1,0,0) = (1,0,1)$

$T(0,1,0) = (1,1,2)$

$T(0,0,1) = (0,1,1)$

but since $(1,1,2) = (1,0,1)+(0,1,1)$

$\therefore R(T) = \text{span}\{(1,0,1),(0,1,1)\}$

$$\mathrm{rank}(T) = 2.$$

f) Null space :

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (0,0,0) \quad : \text{zero element of } \mathbb{R}^3.$$

$$\Rightarrow (a+b, a-b, a) = (0,0,0)$$

$$\Rightarrow \boxed{a = b = 0}$$

$$\therefore N(T) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a = b = 0 \right\}$$

$$= \mathrm{span}\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

$$\therefore \mathrm{nullity}(T) = 2.$$

Range space :

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1,1,1)$$

$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (1,-1,0)$$

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = (0,0,0)$$

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = (0,0,0)$$

$$\therefore R(T) = \mathrm{span}\{(1,1,1), (1,-1,0)\}$$

$$\mathrm{rank}(T) = 2.$$

5// $T(x,y,z) = (3x+y, x+z, x-z)$

$$T(1,0,0) = (3,1,1)$$

$$T(0,1,0) = (1,0,0)$$

$T(0,0,1) = (0, 1, -1)$

$\therefore$ The corresponding matrix is $\begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$.

6) $T(x,y,z) = (-2x+3z,\ x+2y-z)$

$T(1,1,1) = (1, 2) = \boxed{0}(2,1) + \boxed{1}(1,2)$

$T(1,1,-1) = (-5, 4) = \boxed{\frac{-14}{3}}(2,1) + \boxed{\frac{13}{3}}(1,2)$

$T(1,-1,1) = (1,-2) = \boxed{\frac{4}{3}}(2,1) + \boxed{\frac{-5}{3}}(1,2)$

$\alpha(2,1) + \beta(1,2) = (-5, 4)$

$\Rightarrow \begin{cases} 2\alpha + \beta = -5 \\ \alpha + 2\beta = 4 \end{cases} \Rightarrow \begin{array}{r} 4\alpha + 2\beta = -10 \\ -\quad \alpha + 2\beta = 4 \\ \hline 3\alpha = -14 \end{array}$

$\Rightarrow \alpha = -\frac{14}{3}$

$\beta = -5 + \frac{28}{3}$

$\beta = \frac{13}{3}$

$\begin{cases} 2\alpha + \beta = 1 \\ \alpha + 2\beta = -2 \end{cases} \Rightarrow \begin{array}{r} 4\alpha + 2\beta = 2 \\ -\quad \alpha + 2\beta = -2 \\ \hline 3\alpha = 4 \end{array}$

$\alpha = 4/3$

$\beta = 1 - 2\alpha = 1 - \frac{8}{3} \Rightarrow \beta = -\frac{5}{3}$

$\therefore$ The corresponding matrix is

$\begin{pmatrix} 0 & -14/3 & 4/3 \end{pmatrix}$

$$\begin{pmatrix} 0 & -14/3 & 4/3 \\ 1 & 13/3 & -5/3 \end{pmatrix}$$

or $$\frac{1}{3}\begin{pmatrix} 0 & -14 & 4 \\ 3 & 13 & -5 \end{pmatrix}.$$