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	CO1	CO2	CO3	CO4	CO5
Q1			✓		
Q2			✓		
Q3			✓		
Q4			✓		
Q5			✓		
Q6			✓		
Q7			✓		
Q8			✓		

Tutorial Sheet 6 Solutions

1. If $\lambda = 0$ is an eigenvalue of A, then show that A is singular.
Solution: $\lambda \in \mathbb{C}$ is an eigenvalue of the matrix A if there exists a nonzero vector x such that $Ax = \lambda x$. Since $\lambda = 0$ is an eigenvalue of A, that means $Ax = 0x \Rightarrow Ax = 0$. But $Ax = 0$ represents a homogeneous system of equations which has a nonzero solution, which is possible if and only if when A is singular matrix i.e., $\det A = 0$.

OR

The Characteristic equation of A is given by $\det(A - \lambda I_n) = 0$. Eigenvalues are the roots of characteristic equation. Since $\lambda = 0$ is an eigenvalue of A it will satisfy its characteristic equation, i.e., $\det(A - 0I_n) = 0 \Rightarrow \det(A) = 0$.

2. Prove that

(a) Similar matrices have the same characteristic polynomial.
Solution: Suppose matrices A and B are similar i.e., there exists a non-singular matrix X such that $B = XAX^{-1}$. We need to show that $P_A(\lambda) = P_B(\lambda)$.

Now

$$\begin{aligned} P_B(\lambda) &= |\lambda I_n - B| = |\lambda I_n - XAX^{-1}| = |\lambda XX^{-1} - XAX^{-1}| \\ &= |X(\lambda I_n - A)X^{-1}| = |X||\lambda I_n - A||X^{-1}| \\ &= |\lambda I_n - A| \quad (\text{since } |X^{-1}| = \frac{1}{|X|}). \end{aligned}$$

(b) Similar matrices have the same eigenvalues.
Solution: Eigenvalues are the roots of characteristic polynomial. Since similar matrices have the same characteristic polynomial so they have the same eigenvalues.

3. Let $A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$. Then find

(a) The characteristic polynomial of A.
Solution: The characteristic polynomial of A is given by

$$P_A(\lambda) = |\lambda I_n - A| = \left| \begin{bmatrix} \lambda & -2 & 1 \\ -2 & \lambda - 3 & 2 \\ 1 & 2 & \lambda \end{bmatrix} \right| = \lambda^3 - 3\lambda^2 - 9\lambda - 5.$$

(b) The eigenvalues of A.
Solution: Eigenvalues are the roots of characteristic polynomial i.e.,

$$\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0 \;\Rightarrow\; (\lambda+1)^2(\lambda - 5) = 0 \;\Rightarrow\; \lambda = 5, -1, -1.$$

Algebraic multiplicity of the eigenvalue $\lambda = 5$ is 1, and of the eigenvalue $\lambda = -1$ is 2.

(c) The corresponding eigenvectors.
Solution: Eigenvectors corresponding to $\lambda = 5$:
We need to find non-zero vectors $x = [x_1 \; x_2 \; x_3]^t$ such that $Ax = 5x \;\Rightarrow\; (A - 5I_n)x = 0$. This implies $\begin{bmatrix} -5 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. We use row echelon form to solve the above system. Row echelon form is for the system is

$$\begin{bmatrix} -5 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & -2 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2/5 & 1/5 \\ 2 & -2 & -2 \\ -1 & -2 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2/5 & 1/5 \\ 0 & -6/5 & -12/5 \\ 0 & -12/5 & -24/5 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & -2/5 & 1/5 \\ 0 & -6/5 & -12/5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2/5 & 1/5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $R = \begin{bmatrix} 1 & -2/5 & 1/5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Then $(A - 5I_n)x = 0$ if and only if $Rx = 0$. This implies

$$\begin{aligned} & 5x_1 - 2x_2 + x_3 = 0 \quad \text{and} \quad x_2 + 2x_3 = 0 \\ & \Rightarrow x_2 = -2x_3 \quad \text{and} \quad 5x_1 = 2x_2 - x_3 = -5x_3 \\ & \Rightarrow x_1 = -x_3. \end{aligned}$$

Thus the general solution of the system $(A - 5I_n) = 0$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

Eigenvector corresponding to $\lambda = 5$ is $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

Geometric multiplicity of $\lambda = 5$ is 1.

<u>Eigenvectors corresponding to $\lambda = -1$:</u>

We need to find non-zero vectors $x = [x_1 \; x_2 \; x_3]^t$ such that $Ax = -1x \Rightarrow (A+I_n)x = 0$. This implies $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Again, we use row echelon form to solve the above system. Row echelon form for the system is

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This implies that

$$x_1 + 2x_2 - x_3 = 0 \quad \Rightarrow \quad x_1 = -2x_2 + x_3.$$

Thus the general solution of the system $(A + I_n) = 0$ is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Eigenvectors corresponding to $\lambda = -1$ are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Geometric multiplicity of the eigenvalue $\lambda = -1$ is 2.

4. Prove that the eigenvalues of a triangular matrix are the entries on its main diagonal.
   Solution: Let A be an $n \times n$ upper triangular matrix of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n-1} & a_{1n} \\ 0 & a_{22} & \dots & a_{2n-1} & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix}.$$

Then the characteristic polynomial of A is given by

$$\begin{aligned}\det(A-\lambda I_n) &= \det\left(\begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n-1} & a_{1n} \\ 0 & a_{22}-\lambda & \dots & a_{2n-1} & a_{2n} \\ 0 & 0 & a_{33}-\lambda & \dots & a_{3n} \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & a_{nn}-\lambda \end{bmatrix}\right) \\ &= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda).\end{aligned}$$

Therefore the eigenvalues of A are the diagonal entries of A.

A proof for a lower triangular matrix follows similarly.

5. Show that the following matrices A, B and C are diagonalizable. Also, find invertible matrices S_1, S_2 and S_3 such that $S_1^{-1}AS_1, S_2^{-1}BS_2$ and $S_3^{-1}CS_3$ are all diagonal matrices.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & -2 \\ 0 & 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution: Hint: For each of the above matrices follow the following steps.

Step-1: Find all eigenvalues of the matrix.

Step-2: Find all linearly independent eigenvectors corresponding to each eigenvalue.

Step-3: Form the desired matrix S whose columns are nothing but the linearly independent eigenvectors.

6. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Show that $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$ and $\text{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Solution:

(a) We know that the eigenvalues are the roots of characteristic polynomial.

$$P_A(\lambda) = |\lambda I_n - A| \quad \Rightarrow \quad P_A(0) = |-A| = (-1)^n|A|. \tag{0.1}$$

Since $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A, characteristic polynomial can be written as

$$\begin{aligned} &P_A(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_n) \\ &\Rightarrow P_A(0) = (-\lambda_1)(-\lambda_2)\dots(-\lambda_n) \\ &= (-1)^n\lambda_1, \lambda_2, \dots, \lambda_n. \end{aligned} \tag{0.2}$$

From (0.1) and (0.2) we get, $|A| = \lambda_1, \lambda_2, \dots, \lambda_n$.

(b) Eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are roots of n-th degree characteristic polynomial. Now use the fact that sum of the roots of a polynomial is given by the coefficient of the $-\lambda^{n-1}$ term. [Do it yourself.]

7. Let A be 2×2 matrix such that trace$(A) = 4$ and $\det(A) = 5$. Find the eigenvalues of A.
Solution: Let λ_1 and λ_2 be two eigenvalues of A. Then $\lambda_1 + \lambda_2 = 4$ and $\lambda_1 \lambda_2 = 5$. The characteristic equation of A is given by

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0 \Rightarrow \lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda = \frac{4 \pm \sqrt{(4)^2 - 20}}{2} \Rightarrow \lambda = 2 \pm i.$$

8. A 3×3 matrix A has characteristic polynomial $\lambda(\lambda - 1)(\lambda + 2)$. What is the characteristic polynomial of A^2?
Solution: Given $P_A(\lambda) = \lambda(\lambda - 1)(\lambda + 2)$. Eigenvalues of A are given by

$$P_A(\lambda) = 0 \Rightarrow \lambda(\lambda - 1)(\lambda + 2) = 0 \Rightarrow \lambda = 0, 1, -2.$$

Eigenvalues of A^2 are $0, 1, 4$, since for any $\lambda \in \mathbb{C}$ and nonzero x, we have

$$Ax = \lambda x \ \Rightarrow \ A^2 x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x.$$

Therefore, characteristic polynomial of A^2 is $\lambda(\lambda - 1)(\lambda - 4)$.