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**Tutorial Sheet 9 Solution**

1) Solve the following exact/reducible to exact ODEs:

- (a)  $2xye^{x^2} dx + e^{x^2} dy = 0, \quad y(0) = 2;$
- (b)  $\cos(x + y)dx + (3y^2 + 2y + \cos(x + y))dy = 0;$
- (c)  $(1 + 2x) \cos y dx + \sec y dy = 0;$
- (d)  $3x^2 y dx + 4x^3 dy = 0.$

**Solutions:**

(a) Comparing given ODE with  $M(x, y)dx + N(x, y)dy = 0$ , we get

$$M(x, y) = 2xye^{x^2} \quad \text{and} \quad N(x, y) = e^{x^2}.$$

$$\frac{\partial M}{\partial y} = 2xe^{x^2} = \frac{\partial N}{\partial x}$$

Clearly given ODE is exact. In order to find a solution of the given ODE, we must find a function  $F(x, y)$  such that

$$\frac{\partial F}{\partial x} = M(x, y) = 2xye^{x^2}, \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y) = e^{x^2}.$$

From the first expression we get,

$$F(x, y) = \int 2xye^{x^2} dx + \phi(y) \Rightarrow F(x, y) = ye^{x^2} + \phi(y).$$

From the second expression we get,

$$\frac{\partial F}{\partial y} = N(x, y) \Rightarrow e^{x^2} + \phi'(y) = e^{x^2} \Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = c_0,$$

where  $c_0$  is an arbitrary constant. So,

$$F(x, y) = ye^{x^2} + c_0.$$

Thus a one-parameter family of solution is given by

$$F(x, y) = c_1 \Rightarrow ye^{x^2} = c_1 - c_0 = c.$$

Use initial Condition

$$y(0) = 2 \Rightarrow c = 2 \Rightarrow y = 2e^{-x^2}.$$

(b) Here,

$$M(x, y) = \cos(x + y) \quad \text{and} \quad N(x, y) = 3y^2 + 2y + \cos(x + y).$$

$$\frac{\partial M}{\partial y} = -\sin(x + y) = \frac{\partial N}{\partial x}$$

Clearly given ODE is exact. In order to find solution of given ODE, we must find function  $F(x, y)$  such that

$$\frac{\partial F}{\partial x} = M(x, y) = \cos(x + y), \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y) = 3y^2 + 2y + \cos(x + y).$$

From the first expression we get,

$$F(x, y) = \int \cos(x + y) dx + \phi(y) \Rightarrow F(x, y) = \sin(x + y) + \phi(y).$$

From the second expression we get,

$$\begin{aligned} \frac{\partial F}{\partial y} = N(x, y) &\Rightarrow \cos(x + y) + \phi'(y) = 3y^2 + 2y + \cos(x + y) \\ \Rightarrow \phi'(y) = 3y^2 + 2y &\Rightarrow \phi(y) = y^3 + y^2 + c_0, \end{aligned}$$

where  $c_0$  is an arbitrary constant. So,

$$F(x, y) = \sin(x + y) + y^3 + y^2 + c_0.$$

Thus a one-parameter family of solution is given by

$$\begin{aligned} F(x, y) = c_1 &\Rightarrow \sin(x + y) + y^3 + y^2 = c_1 - c_0 = c. \\ \Rightarrow \sin(x + y) + y^3 + y^2 &= c. \end{aligned}$$

(c)

$$M(x, y) = (1 + 2x) \cos y \quad \text{and} \quad N(x, y) = \sec y$$

$$\frac{\partial M}{\partial y} = -(1 + 2x) \sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

hence given ODE is not exact.

$$\frac{N_x - M_y}{M} = \frac{(1 + 2x) \sin y}{(1 + 2x) \cos y} = \frac{\sin y}{\cos y},$$

$$\text{so I. F.} = \int e^{\frac{\sin y}{\cos y}} dy = \sec y.$$

Multiply given equation by I. F., we get

$$(1 + 2x) dx + \sec^2 y dy = 0.$$

Now given ODE is exact. Thus

$$F(x, y) = \int (1 + 2x)dx + \phi(y) = x + x^2 + \phi(y)$$

$$\text{Now } \frac{\partial F}{\partial y} = N(x, y) \Rightarrow \phi'(y) = \sec^2 y \Rightarrow \phi(y) = \tan y + c_0,$$

where  $c_0$  is an arbitrary constant. So,

$$F(x, y) = c_1 \Rightarrow x + x^2 + \tan y + c_0 = c_1 \Rightarrow \tan y = -x - x^2 + c.$$

(d)

$$M(x, y) = 3x^2y \quad \text{and} \quad N(x, y) = 4x^3.$$

$$\frac{\partial M}{\partial y} = 3x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 12x^2.$$

Clearly given ODE is not exact.

$$\frac{N_x - M_y}{M} = \frac{12x^2 - 3x^2}{3x^2y} = \frac{3}{y},$$

$$\text{so I. F.} = \int e^{\frac{3}{y}dy} = y^3.$$

Multiply given equation by I. F., we get

$$3x^2y^4dx + 4x^3y^3dy = 0.$$

This ODE is exact.

$$F(x, y) = \int 3x^2y^4dx + \phi(y) \Rightarrow F(x, y) = x^3y^4 + \phi(y)$$

Now

$$\frac{\partial F}{\partial y} = N(x, y) \Rightarrow 4x^3y^3 + \phi'(y) = 4x^3y^3 \Rightarrow \phi(y) = c_0,$$

where  $c_0$  is an arbitrary constant. So,

$$F(x, y) = c_1 \Rightarrow x^3y^4 + c_0 = c_1 \Rightarrow x^3y^4 = c_1 - c_0 = c.$$

2) Solve the following linear/reducible to linear ODEs:

(a)  $\frac{dy}{dx} + 3x^2y = x^2, \quad y(0) = 2;$

(b)  $y^2dx + (3xy - 1)dy = 0;$

(c)  $\frac{dy}{dx} + y = f(x), \quad y(0) = 0,$  where  $f(x) = \begin{cases} 2 & 0 \leq x < 1, \\ 0 & x \geq 1. \end{cases},$

(d)  $dy + (4y - 8y^{-3})xdx = 0.$

**Solution:**

(a) Given ODE is linear in  $y$ . Compare with

$$\frac{dy}{dx} + P(x)y = Q(x),$$

we get  $P(x) = 3x^2$  and  $Q(x) = x^2$ .

$$I. F. = e^{\int p(x)dx} = e^{\int 3x^2 dx} = e^{x^3}.$$

Solution is

$$y \times I.F. = \int (Q(x) \times I.F.) dx + c,$$

where  $c$  is an arbitrary constant.

$$y \times e^{x^3} = \int x^2 e^{x^3} dx + c \Rightarrow y \times e^{x^3} = \frac{1}{3} \times e^{x^3} + c$$

Now use initial condition

$$y(0) = 2 \Rightarrow 2 = \frac{1}{3} + c \Rightarrow c = \frac{5}{3}.$$

Solution is

$$y(x) = \frac{5}{3}e^{-x^3} + \frac{1}{3}.$$

(b) This ODE is linear in  $x$ . Rewrite as

$$\frac{dx}{dy} + \frac{3}{y}x = \frac{1}{y^2}.$$

$$I. F. = e^{\int p(y)dy} = e^{\int \frac{3}{y} dy} = y^3.$$

Solution is

$$x \times y^3 = \int \frac{1}{y^2} y^3 dy + c \Rightarrow x \times y^3 = \frac{y^2}{2} + c,$$

where  $c$  is an arbitrary constant.

(c) Here,

$$I.F. = e^{\int P(x)dx} = e^{\int 1 dx} = e^x.$$

Solution is

$$y \times I.F. = \int f(x) \times I.F. dx + c,$$

where  $c$  is an arbitrary constant.  $f(x) = 2$  when  $0 \leq x < 1$ , i.e.,

$$y.e^x = \int 2e^x dx + c \Rightarrow y(x) = 2 + ce^{-x}.$$

Use initial condition

$$y(0) = 0 \Rightarrow 0 = 2 + c \Rightarrow c = -2.$$

That is,

$$y(x) = 2(1 - e^{-x}).$$

Now for  $x \geq 1$ ,  $f(x) = 0$ , then solution is

$$y \cdot e^x = c_1 \Rightarrow y = c_1 e^{-x},$$

where  $c_1$  is an arbitrary constant. To determine  $c_1$ , use continuity of  $y(x)$  at  $x = 1$ . i.e.,

$$\lim_{x \rightarrow 1^+} y(x) = \lim_{x \rightarrow 1^-} y(x) \Rightarrow c_1 e^{-1} = 2(1 - e^{-1}) \Rightarrow c = 2(e - 1).$$

So,  $y(x) = 2(e - 1)e^{-x}$ . Hence,

$$y(x) = \begin{cases} 2(1 - e^{-x}) & 0 \leq x < 1, \\ 2(e - 1)e^{-x} & x \geq 1. \end{cases}$$

(d)

$$\frac{dy}{dx} + 4xy = 8xy^{-3}$$

Multiply by  $y^3$ , we get

$$y^3 \frac{dy}{dx} + 4xy^4 = 8x.$$

Take,

$$y^4 = v \Rightarrow y^3 \frac{dy}{dx} = \frac{1}{4} \frac{dv}{dx}.$$

Now transformed ODE is

$$\frac{1}{4} \frac{dv}{dx} + 4xv = 8x$$

This is linear in  $v$ . Solution is

$$v = 2 + ce^{-8x^2} \Rightarrow y^4 = 2 + ce^{-8x^2}.$$

- 3) Under what conditions for the constants  $a, b, k, l$ , is  $(ax + by)dx + (kx + ly)dy = 0$  exact? Solve the exact ODE.

**Solution:** For exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow b = k.$$

For solution of exact DE

$$F(x, y) = \int (ax + by)dx + \phi(y) \Rightarrow F(x, y) = \frac{ax^2}{2} + bxy + \phi(y).$$

Now to determine  $\phi(y)$ , use

$$\frac{\partial F}{\partial y} = N(x, y) \Rightarrow bx + \phi'(y) = kx + ly \Rightarrow \phi(y) = \frac{ly^2}{2} + c_0.$$

$$\text{Thus } F(x, y) = c_1 \Rightarrow \frac{ax^2}{2} + bxy + \frac{ly^2}{2} = c.$$

- 4) Does the IVP  $(x - 2)\frac{dy}{dx} = y$ ;  $y(2) = 1$  have a solution? Justify your answer.

**Solution:** No solution.

5) Show that existence and uniqueness theorem guarantees the existence of a unique solution of the IVP-

(a)  $\frac{dy}{dx} = e^{2y}; y(0) = 0.$

(b)  $\frac{dy}{dx} = y^{4/3}; y(x_0) = y_0.$

**Solutions:**

(a) Consider a rectangle  $R : |x| \leq a, |y| \leq b$ . Clearly the function  $e^{2y}$  is continuous in  $R$  and  $|e^{2y}| \leq e^{2b} = k$  in  $R$ , so by existence theorem there exists a solution of given IVP in the interval

$$|x| \leq \alpha, \text{ where } \alpha = \min \left( a, \frac{b}{k} \right) = \min \left( a, \frac{b}{e^{2b}} \right).$$

Also,

$$\frac{\partial f}{\partial y} = \frac{\partial e^{2y}}{\partial y} = 2e^{2y},$$

is bounded in  $R$ , so by uniqueness theorem, there exists a unique solution of IVP in above defined interval,  $|x| \leq \alpha$ . where,

$$\alpha = \min \left( a, \frac{b}{e^{2b}} \right).$$

Consider  $F(b) = \frac{b}{e^{2b}} \Rightarrow F'(b) = \frac{1-2b}{e^{2b}}$ . The maximum value of  $F(b)$  occurs at  $b = \frac{1}{2}$  and  $F(1/2) = \frac{1}{2e}$ . Now if  $a \geq F(b)$ , then  $\alpha = \min(a, F(b)) = F(b) \leq \frac{1}{2e}$  for all  $b > 0$ . If  $a < \frac{1}{2e} \Rightarrow \alpha < \frac{1}{2e}$ . For  $b = \frac{1}{2}, a \geq \frac{1}{2e}, \alpha = \frac{1}{2e}$ . Thus, in any case  $\alpha \leq \frac{1}{2e}$ , i.e.,  $|x| \leq \frac{1}{2e}$ .

(b) Proceed like (a) part.